

New definitions of contexts and free variables for λ -calculi with explicit substitutions

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The aim of this paper is to give more convenient definitions of substitution and α -conversion. I propose the calculus $\lambda\alpha$ with the following characteristic features:

1. Named variables;
2. Explicit substitutions;
3. Explicit weakening;
4. Explicit renaming of bound variables (there is an “ α -reduction”);
5. It is possible that $x \in FV(\lambda x.A)$.

I believe that it is much more convenient to write substitution on the left: $S \circ A$ instead of $A[S]$ and

$$\begin{array}{ll}
 \lambda x.S \circ A & \text{instead of } \lambda x.(A[S]) \\
 S \circ \lambda x.A & \text{instead of } (\lambda x.A)[S] \\
 \lambda x.S_2 \circ S_1 \circ A & \text{instead of } \lambda x.(A[S_1][S_2]) \\
 S_2 \circ \lambda x.S_1 \circ A & \text{instead of } (\lambda x.A[S_1])[S_2] \\
 S_2 \circ S_1 \circ \lambda x.A & \text{instead of } (\lambda x.A)[S_1][S_2]
 \end{array}$$

And so on. Far fewer parentheses. And very close to notation of category theory. If we have composition of substitutions (not in my calculus), then

$$\lambda x.(S_2 \circ S_1) \circ A \text{ means } \lambda x.(A[S_1 \circ S_2])$$

From the point of view of category theory, the symbol \circ in $S \circ A$ and $S_2 \circ S_1$

is the same composition. It is convenient also to write

$S \circ AB_1 \dots B_n$ instead of $S \circ (AB_1 \dots B_n)$

because we write

$\lambda x.AB_1 \dots B_n$ instead of $\lambda x.(AB_1 \dots B_n)$

Then we need parenthesis only to separate parts of application.

There are four kinds of substitutions in $\lambda\alpha$:

1. $[B/x]$
2. Wx (explicit weakening, corresponds to \uparrow)
3. $\{yx\}$ (explicit renaming, very roughly corresponds to Pitts's (yx))
4. S_x (corresponds to $\uparrow\uparrow S$)

Then “ α -reduction” looks as follows:

$\lambda x.A \rightarrow \lambda y.\{yx\} \circ A$

Of course, we need restrictions to avoid infinite chains of renaming:

$\lambda x.A \rightarrow \lambda y.\{yx\} \circ A \rightarrow \lambda z.\{zy\} \circ \{yx\} \circ A \rightarrow \dots$

To give such restrictions we have to define FV for terms with explicit substitutions. It turns that $FV(A)$ in general is not a set, but has a more complicated structure (it is a set if A does not contain explicit substitutions).

I tried to create a calculus with named variables so closed to calculi with De Bruijn indices as possible. The main problem was to define FV .

Example 0.1.

$$\begin{aligned} & (\lambda x.x) y \\ & \rightarrow [y/x] \circ x \\ & \rightarrow y \end{aligned}$$

Example 0.2.

$$\begin{aligned} & (\lambda x.\lambda y.x) z \\ & \rightarrow [z/x] \circ \lambda y.x \\ & \rightarrow \lambda y.[z/x]_y \circ x \\ & \rightarrow \lambda y.Wy \circ [z/x] \circ x \\ & \rightarrow \lambda y.Wy \circ z \\ & \rightarrow \lambda y.z \end{aligned}$$

corresponds to “De Bruijn’s calculation”

$$\begin{aligned}
& (\lambda \lambda \underline{1}[\uparrow]) z \\
& \rightarrow (\lambda \underline{1}[\uparrow])[z/] \\
& \rightarrow \lambda(\underline{1}[\uparrow][\uparrow(z/)]) \\
& \rightarrow \lambda(\underline{1}[z/][\uparrow]) \\
& \rightarrow \lambda(z[\uparrow])
\end{aligned}$$

Example 0.3.

$$\begin{aligned}
& (\lambda x. \lambda y. x) y \\
& \rightarrow [y/x] \circ \lambda y. x \\
& \rightarrow \lambda y. [y/x]_y \circ x \\
& \rightarrow \lambda y. W y \circ [y/x] \circ x \\
& \rightarrow \lambda y. W y \circ y \\
& \rightarrow \lambda z. \{zy\} \circ W y \circ y \quad (\alpha) \\
& \rightarrow \lambda z. W z \circ y \\
& \rightarrow \lambda z. y
\end{aligned}$$

It turns that the rightmost occurrence of y in $\lambda y. W y \circ y$ is free, hence $y \in FV(\lambda y. W y \circ y)$. The terms $\lambda y. W y \circ y$ and $\lambda z. W z \circ y$ both correspond to $\lambda(y[\uparrow])$. They are α -equivalent in this sense and the following reductions

$$\begin{aligned}
& \lambda y. W y \circ y \\
& \rightarrow \lambda z. \{zy\} \circ W y \circ y \quad (\alpha) \\
& \rightarrow \lambda z. W z \circ y
\end{aligned}$$

rename bound variables, but do not change the free variable. The α -reduction can be written as follows

$$\lambda x. A \rightarrow \lambda y. \{yx\} \circ A \quad (x \in FV(\lambda x. A) \ \& \ y \notin FV(\lambda x. A))$$

1 Contexts, terms, and substitutions of the calculus $\lambda\alpha$

Convention 1.1. The symbols $x, y, z \dots$ are *variables*. The symbols x, y, z range over variables. The inequality $x \neq y$ means that x and y denote different variables.

Definition 1.2. A *global context* is a possibly empty, finite set of variables.

Definition 1.3. A *local context* is a possibly empty, finite list of variables with multiplicity (i.e., repetitions are permitted).

Example 1.4. The list x, x, y is a local context.

Definition 1.5. A *context* is a pair G, L , where G is a global context and L is a local context. The symbols Γ, Δ, Σ range over contexts. If Γ is G, L , then $x \in \Gamma$ is shorthand for $x \in G \vee x \in L$.

Example 1.6. $\{x, z\}, x, x, y$ is a context, where $\{x, z\}$ is a global context and x, x, y is a local context.

Convention 1.7. If Γ is a context, composed from G and L , then we denote by Γ, x the context, composed from G and L, x .

Definition 1.8. *Terms* and *substitutions* of the calculus $\lambda\alpha$ are shown in Figure 1. A *judgement* is an expression of the form $\Gamma \vdash A$ or of the form $\Gamma \vdash S \triangleright \Delta$. Inference rules for judgements are shown in Figure 1 ($G \vdash x$ is shorthand for $G, nil \vdash x$, where nil is the empty list). A term A is *well-formed* iff $\Gamma \vdash A$ is derivable for some Γ .

Convention 1.9.

$S \circ AB_1 \dots B_k$	is shorthand for	$S \circ (AB_1 \dots B_k)$
$\lambda x. S \circ A$	is shorthand for	$\lambda x. (S \circ A)$
$S \circ \lambda x. A$	is shorthand for	$S \circ (\lambda x. A)$
$S_1 \circ \dots \circ S_k \circ A$	is shorthand for	$S_1 \circ (\dots \circ (S_{k-1} \circ (S_k \circ A)) \dots)$
$S_{xy\dots z}$	is shorthand for	$(\dots ((S_x)_y) \dots)_z$

In the similar calculus with types the judgement $\Gamma \vdash A : T$ means “the rightmost occurrence of x in Γ has type T ”. Hence, we can derive

$\{x : \mathbb{N}, y : \mathbb{R}\}, x : \mathbb{R} \vdash x : \mathbb{R}$

$\{x : \mathbb{N}, y : \mathbb{R}\}, x : \mathbb{R} \vdash y : \mathbb{R}$

but we can not derive

$\{x : \mathbb{N}, y : \mathbb{R}\}, x : \mathbb{R} \vdash x : \mathbb{N}$.

Syntax. $\Lambda\alpha$ is the set of terms inductively defined by the following BNF:

$$\begin{aligned} x &::= x \mid y \mid z \mid \dots && \text{(Variables)} \\ A, B &::= x \mid AB \mid \lambda x. A \mid S \circ A && \text{(Terms)} \\ S &::= [B/x] \mid Wx \mid \{yx\} \mid S_x && \text{(Substitutions)} \end{aligned}$$

Inference rules.

$$\begin{aligned} R1 & \quad G \vdash x && (x \in G) \\ R2 & \quad \Gamma, x \vdash x \\ R3 & \quad \frac{\Gamma \vdash x}{\Gamma, y \vdash x} && (x \neq y) \\ R4 & \quad \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash AB} \\ R5 & \quad \frac{\Gamma, x \vdash A}{\Gamma \vdash \lambda x. A} \\ R6 & \quad \frac{\Gamma \vdash S \triangleright \Delta \quad \Delta \vdash A}{\Gamma \vdash S \circ A} \\ R7 & \quad \frac{\Gamma \vdash B}{\Gamma \vdash [B/x] \triangleright \Gamma, x} \\ R8 & \quad \Gamma, x \vdash Wx \triangleright \Gamma \\ R9 & \quad \Gamma, y \vdash \{yx\} \triangleright \Gamma, x \\ R10 & \quad \frac{\Gamma \vdash S \triangleright \Delta}{\Gamma, x \vdash S_x \triangleright \Delta, x} \end{aligned}$$

Figure 1: Terms and substitutions

Example 1.10.

$$\frac{\frac{\frac{\emptyset, x, x \vdash x}{\emptyset, x \vdash \lambda x.x}}{\emptyset \vdash \lambda x.\lambda x.x}}$$

Example 1.11.

$$\frac{\frac{\frac{\frac{\frac{\emptyset, x \vdash x}{\emptyset, x, y \vdash x} \quad \emptyset, x, y \vdash y}{\emptyset, x, y \vdash xy}}{\emptyset, x \vdash \lambda y.xy}}{\emptyset \vdash \lambda x.\lambda y.xy}}$$

Example 1.12.

$$\frac{\frac{\frac{\frac{\{x\} \vdash x}{\{x\}, y \vdash x} \quad \{x\}, y \vdash y}{\{x\}, y \vdash xy}}{\{x\} \vdash \lambda y.xy}}$$

Example 1.13.

$$\frac{\frac{\frac{\{x\}, x \vdash Wx \triangleright \{x\} \quad \{x\} \vdash x}{\{x\}, x \vdash Wx \circ x}}{\{x\} \vdash \lambda x.Wx \circ x}}$$

Proposition 1.14. *The following rules are admissible:*

$$\frac{\Gamma \vdash A}{\Gamma, x \vdash Wx \circ A} \quad \frac{\Gamma, x \vdash A}{\Gamma, y \vdash \{yx\} \circ A} \quad \frac{\Gamma, x \vdash A \quad \Gamma \vdash B}{\Gamma \vdash [B/x] \circ A}$$

Proof.

$$\frac{\frac{\Gamma, x \vdash Wx \triangleright \Gamma \quad \Gamma \vdash A}{\Gamma, x \vdash Wx \circ A} \quad \frac{\frac{\Gamma, y \vdash \{yx\} \triangleright \Gamma, x \quad \Gamma, x \vdash A}{\Gamma, y \vdash \{yx\} \circ A} \quad \frac{\frac{\Gamma \vdash B}{\Gamma \vdash [B/x] \triangleright \Gamma, x} \quad \Gamma, x \vdash A}{\Gamma \vdash [B/x] \circ A}}$$

□

Example 1.15. A judgement of the form $\Gamma \vdash \lambda x.Wy \circ A$ is not derivable if $x \neq y$.

$$\frac{\frac{\Gamma, y \vdash Wy \triangleright \Gamma \quad \Gamma \vdash A}{\Gamma, y \vdash Wy \circ A}}{\Gamma \vdash \lambda x.Wy \circ A} (?)$$

Hence, a term of the form $\lambda x.Wy \circ M$ is not well-formed if $x \neq y$.

Example 1.16. A term of the form $(Wx \circ A)(Wy \circ B)$ is not well-formed if $x \neq y$.

$$\frac{\frac{\Gamma, x \vdash Wx \triangleright \Gamma \quad \Gamma \vdash A}{\Gamma, x \vdash Wx \circ A} \quad \frac{\Delta, y \vdash Wy \triangleright \Delta \quad \Delta \vdash B}{\Delta, y \vdash Wy \circ B}}{? \vdash (Wx \circ A)(Wy \circ B)}$$

Lemma 1.17 (Generation lemma).

Each derivation of $G \vdash x$ is an application of the rule R1.

Each derivation of $\Gamma, x \vdash x$ is an application of the rule R2.

Each derivation of $\Gamma, y \vdash x$ (where $x \neq y$) is an application of the rule R3 to some derivation of $\Gamma \vdash x$.

Each derivation of $\Gamma \vdash AB$ is an application of the rule R4 to some derivations of $\Gamma \vdash A$ and $\Gamma \vdash B$.

Each derivation of $\Gamma \vdash \lambda x.A$ is an application of the rule R5 to some derivation of $\Gamma, x \vdash A$.

Each derivation of $\Gamma \vdash S \circ A$ is an application of the rule R6 to some derivations of $\Gamma \vdash S \triangleright \Delta$ and $\Delta \vdash A$ for some Δ .

Each derivation of $\Gamma \vdash [B/x] \triangleright \Delta$ is an application of the rule R7 to some derivation of $\Gamma \vdash B$, where Δ is Γ, x .

Each derivation of $\Delta \vdash Wx \triangleright \Gamma$ is an application of the rule R8, where Δ is Γ, x .

Each derivation of $\Delta \vdash \{yx\} \triangleright \Sigma$ is an application of the rule R9, where Δ is Γ, y and Σ is Γ, x for some Γ .

Each derivation of $\Sigma \vdash S_x \triangleright \Psi$ is an application of the rule R10 to some derivation of $\Gamma \vdash S \triangleright \Delta$, where Σ is Γ, x and Ψ is Δ, x .

Proof. The proof is straightforward. □

Corollary 1.18. *Subterms of well-formed terms are well-formed.*

Lemma 1.19. $\Gamma \vdash x$ is derivable iff $x \in \Gamma$.

Proof. Induction over the length of local part of Γ . If this length is equal to 0, then Γ has the form G and $G \vdash x$ is derivable iff $x \in G$. If Γ has the form Δ, y then either $x = y$ or $x \neq y$. In the first case $\Gamma \vdash x$ is derivable and $x \in \Gamma$. In the last case $\Delta, y \vdash x$ is derivable iff $\Delta \vdash x$ is derivable and we use the induction hypothesis. □

Proposition 1.20. If a judgement of the form $\Gamma \vdash S \triangleright \Delta$ is derivable, then Δ is uniquely defined for given Γ and S .

Proof. The proof is by induction over the structure of S .

Case 1. S has the form $[B/x]$. Then Δ is Γ, x .

Case 2. S has the form Wx . Then Γ is Δ, x .

Case 3. S has the form $\{yx\}$. Then Γ has the form Σ, y and Δ is Σ, x .

Case 4. S has the form S'_x . By Generation lemma, we can derive $\Sigma \vdash S' \triangleright \Psi$, where Γ is Σ, x and Δ is Ψ, x . By the induction hypothesis, Ψ is uniquely defined for Σ and S' . □

Proposition 1.21. For any derivable judgement, there is a unique derivation. The problem of derivability for judgements is decidable.

Proof. We try to construct a derivation from the bottom up, using Generation lemma and the previous proposition. □

We obtain this result because there are no weakening rules except of $R3$. Now we define a partial order on the set of contexts. We want $FV(A)$ to be “the least” context Γ such that $\Gamma \vdash A$ is derivable. We want $FV(x) = \{x\}$. But $\emptyset, x \vdash x$ is derivable too, hence we want $\{x\} < \emptyset, x$

Definition 1.22. The order \leq is the least partial order on the set of contexts satisfying the following properties:

1. $G, L < G \cup \{x\}, L \quad (x \notin G)$
2. $G, L < (G - \{x\}), x, L$

Example 1.23. $\{z\}, y < \{z, x\}, y < \{z\}, x, y < \{z, x\}, x, y$

Proposition 1.24. $G_1, L_1 \leq G_2, L_2$ iff $L_2 = LL_1$ for some L and $\forall x \in G_1 (x \in G_2 \vee x \in L)$.

Proof. Straightforward. □

Proposition 1.25.

1. $\Gamma, x \leq \Delta, x$ iff $\Gamma \leq \Delta$;
2. $\Gamma, x \leq \Delta$ imply Δ has the form Σ, x ;
3. $\Gamma \leq \Delta, x$ imply Γ has the form Σ, x or Γ has the form G ;
4. $\Gamma \leq G$ imply Γ is also a set.

Proof. Straightforward. □

Theorem 1.26. *If $\Gamma \vdash A$ is derivable and $\Gamma \leq \Sigma$ then $\Sigma \vdash A$ is derivable. If $\Gamma \vdash S \triangleright \Delta$ is derivable and $\Gamma \leq \Sigma$ then $\Sigma \vdash S \triangleright \Psi$ is derivable for some $\Psi \geq \Delta$.*

Proof. Induction over the structure of A and S .

Case 1. A is x . Then $\Gamma \vdash x$ is derivable iff $x \in \Gamma$ (Lemma 1.19). It is easy to prove that $x \in \Gamma$ and $\Gamma \leq \Sigma$ imply $x \in \Sigma$, hence $\Sigma \vdash x$ is derivable.

Case 2. A is $B_1 B_2$. By Generation lemma $\Gamma \vdash B_1$ and $\Gamma \vdash B_2$ are derivable. By induction hypothesis $\Sigma \vdash B_1$ and $\Sigma \vdash B_2$ are derivable, hence $\Sigma \vdash B_1 B_2$ is derivable.

Case 3. A is $\lambda x.B$. By Generation lemma $\Gamma, x \vdash B$ is derivable. $\Gamma \leq \Sigma$ imply $\Gamma, x \leq \Sigma, x$. By induction hypothesis $\Sigma, x \vdash B$ is derivable, hence $\Sigma \vdash \lambda x.B$ is derivable.

Case 4. A is $S \circ B$. By Generation lemma $\Gamma \vdash S \triangleright \Delta$ and $\Delta \vdash B$ are derivable for some Δ . By induction hypothesis $\Sigma \vdash S \triangleright \Psi$ and $\Psi \vdash B$ are derivable for some $\Psi \geq \Delta$, hence $\Sigma \vdash S \circ B$ is derivable.

Case 5. S is $[B/x]$. $\Gamma \vdash [B/x] \triangleright \Gamma, x$ is derivable, hence $\Gamma \vdash B$ is derivable. By induction hypothesis $\Sigma \vdash B$ is derivable, hence $\Sigma \vdash [B/x] \triangleright \Sigma, x$ is derivable. $\Gamma \leq \Sigma$ imply $\Gamma, x \leq \Sigma, x$.

Case 6. S is Wx . Then Γ has the form Δ, x . $\Delta, x \leq \Sigma$ imply Σ has the form Ψ, x . $\Psi, x \vdash Wx \triangleright \Psi$ is derivable. $\Gamma \leq \Sigma$ imply $\Delta \leq \Psi$.

Case 7. S is $\{yx\}$. Then Γ has the form Γ', y and Δ has the form Γ', x . $\Gamma', y \leq \Sigma$ imply Σ has the form Σ', y . Put $\Psi = \Sigma', x$, then $\Sigma \vdash \{yx\} \triangleright \Psi$ is derivable and $\Delta \leq \Psi$.

Case 8. S is S'_x . Then Γ has the form Γ', x and Δ has the form Δ', x where $\Gamma' \vdash S' \triangleright \Delta'$ is derivable. $\Gamma', x \leq \Sigma$ imply Σ has the form Σ', x and $\Gamma' \leq \Sigma'$. By induction hypothesis $\Sigma' \vdash S' \triangleright \Psi'$ is derivable for some $\Psi' \geq \Delta'$. Put $\Psi = \Psi', x$. Then $\Sigma \vdash \{Sx\} \triangleright \Psi$ is derivable and $\Psi \geq \Delta$.

□

Definition 1.27. Γ and Δ are *compatible* iff $\Gamma \leq \Sigma$ and $\Delta \leq \Sigma$ for some Σ .

Proposition 1.28. *Any set G is compatible with any Γ .*

G_1, L_1 and G_2, L_2 are compatible iff $L_1 = LL_2$ or $L_2 = LL_1$ for some L .

If Γ and Δ are compatible, there exists their supremum $\Gamma \sqcup \Delta$.

Proof. $\Gamma \sqcup \Delta$, if exists, can be calculated recursively using the following rules:

$$(\Gamma, x) \sqcup (\Delta, x) = (\Gamma \sqcup \Delta), x$$

$$(\Gamma, x) \sqcup G = (\Gamma \sqcup (G - \{x\})), x$$

$$G \sqcup (\Gamma, x) = ((G - \{x\}) \sqcup \Gamma), x$$

$$G_1 \sqcup G_2 = G_1 \cup G_2$$

□

Example 1.29. $(\{x\}, z) \sqcup \{y, z\} = (\{x\} \cup \{y\}), z = \{x, y\}, z$

$$\begin{aligned}
FV(x) &= \{x\} \\
FV(AB) &= FV(A) \sqcup FV(B) \\
FV(\lambda x.A) &= O_{\lambda x}(FV(A)) \\
FV(Wx \circ A) &= FV(A), x \\
FV([B/x] \circ A) &= FV((\lambda x.A)B) \\
FV(\{yx\} \circ A) &= FV(Wy \circ \lambda x.A) \\
FV(S_x \circ A) &= FV(Wx \circ S \circ \lambda x.A) \\
\\
O_{\lambda x}(\Gamma, x) &= \Gamma \\
O_{\lambda x}(G) &= G - \{x\}
\end{aligned}$$

Figure 2: Free variables

2 Free variables

Definition 2.1. The definition of free variables is shown in Figure 2. $FV(A)$ is not a set, but a context (the least context Γ such that $\Gamma \vdash A$ is derivable).

Example 2.2. $FV(\lambda x.xy) = O_{\lambda x}(FV(xy)) = O_{\lambda x}(FV(x) \sqcup FV(y)) = O_{\lambda x}(\{x\} \cup \{y\}) = O_{\lambda x}(\{x, y\}) = \{y\}$

Example 2.3. $FV(\lambda x.Wx \circ x) = O_{\lambda x}(FV(Wx \circ x)) = O_{\lambda x}(FV(x), x) = O_{\lambda x}(\{x\}, x) = \{x\}$

We see that $x \in FV(\lambda x.Wx \circ x)$. This is strange, but true (see Example 1.13 to understand).

Note that $FV(A)$ is not always defined. For example, $FV(\lambda x.Wy \circ A) = O_{\lambda x}(FV(Wy \circ A)) = O_{\lambda x}(FV(A), y)$ is not defined if $x \neq y$ (but this term is not well-formed).

Lemma 2.4. *If $O_{\lambda x}(\Gamma \sqcup \Delta)$ is defined, then $O_{\lambda x}(\Gamma \sqcup \Delta) = O_{\lambda x}(\Gamma) \sqcup O_{\lambda x}(\Delta)$*

Proof. See the proof of Proposition 1.28. □

Corollary 2.5. *If $O_{\lambda x}(\Gamma)$ is defined and $\Gamma \geq \Delta$, then $O_{\lambda x}(\Delta)$ is defined and $O_{\lambda x}(\Gamma) \geq O_{\lambda x}(\Delta)$.*

Lemma 2.6. *$\Gamma \vdash [B/x] \circ A$ is derivable iff $\Gamma \vdash (\lambda x.A)B$ is derivable.*

Proof.

$$\frac{\frac{\vdots}{\Gamma \vdash B} \quad \vdots}{\Gamma \vdash [B/x] \triangleright \Gamma, x} \quad \frac{\vdots}{\Gamma, x \vdash A} \quad \frac{\vdots}{\Gamma, x \vdash A} \quad \frac{\vdots}{\Gamma \vdash \lambda x.A} \quad \frac{\vdots}{\Gamma \vdash B} \quad \frac{\vdots}{\Gamma \vdash (\lambda x.A)B}$$

□

Lemma 2.7. *$\Gamma \vdash \{yx\} \circ A$ is derivable iff $\Gamma \vdash Wy \circ \lambda x.A$ is derivable.*

Proof.

$$\frac{\Delta, y \vdash \{yx\} \triangleright \Delta, x \quad \frac{\vdots}{\Delta, x \vdash A}}{\Delta, y \vdash \{yx\} \circ A}$$

$$\frac{\Delta, y \vdash Wy \triangleright \Delta \quad \frac{\vdots}{\Delta, x \vdash A} \quad \frac{\vdots}{\Delta \vdash \lambda x.A}}{\Delta, y \vdash Wy \circ \lambda x.A}$$

□

Lemma 2.8. *$\Gamma \vdash S_x \circ A$ is derivable iff $\Gamma \vdash W_x \circ S \circ \lambda x.A$ is derivable.*

Proof.

$$\frac{\frac{\vdots}{\Delta \vdash S \triangleright \Sigma} \quad \vdots}{\Delta, x \vdash S_x \triangleright \Sigma, x} \quad \frac{\vdots}{\Sigma, x \vdash A} \quad \frac{\vdots}{\Delta, x \vdash S_x \circ A}$$

$$\frac{\Delta, x \vdash W_x \triangleright \Delta \quad \frac{\vdots}{\Delta \vdash S \triangleright \Sigma} \quad \frac{\vdots}{\Sigma, x \vdash A} \quad \frac{\vdots}{\Sigma \vdash \lambda x.A}}{\Delta, x \vdash W_x \circ S \circ \lambda x.A}$$

□

Theorem 2.9. *If $\Gamma \vdash A$ is derivable, then $FV(A)$ is defined, $FV(A) \vdash A$ is derivable, and $FV(A) \leq \Gamma$.*

Proof. Induction over

1. the total number of substitutions $[/]$, $\{\}$, and S_x in A ;
2. the length of A .

Note that each step of calculation of $FV(A)$ (shown in Figure 2) decreases the number of $[/]$, $\{\}$, and S_x , or takes a subterm. There are the following seven possible cases.

Case 1. A is x . Then $FV(A) = \{x\}$. If $\Gamma \vdash x$ is derivable, then $x \in \Gamma$ (Lemma 1.19), hence $\{x\} \leq \Gamma$.

Case 2. A is $B_1 B_2$. Then $\Gamma \vdash B_1$ and $\Gamma \vdash B_2$ are derivable by Generation lemma. By induction hypothesis $FV(B_1) \vdash B_1$ and $FV(B_2) \vdash B_2$ are derivable, $FV(B_1) \leq \Gamma$, and $FV(B_2) \leq \Gamma$. Hence $FV(B_1) \sqcup FV(B_2)$ exists (by Proposition 1.28) and $FV(B_1) \sqcup FV(B_2) \leq \Gamma$. Note that $FV(B_1) \sqcup FV(B_2) \vdash B_1$ and $FV(B_1) \sqcup FV(B_2) \vdash B_2$ are derivable by Theorem 1.26, hence $FV(B_1) \sqcup FV(B_2) \vdash B_1 B_2$ is derivable.

Case 3. A is $\lambda x.B$. Then $\Gamma, x \vdash B$ by Generation lemma. By induction hypothesis $FV(B) \vdash B$ is derivable and $FV(B) \leq \Gamma, x$. Hence $FV(B)$ has the form Δ, x or $FV(B)$ has the form G .

In the first case $FV(\lambda x.B) = \Delta$ and $\Delta, x \vdash B$ is derivable (Δ, x is $FV(B)$), hence $FV(\lambda x.B) \vdash \lambda x.B$ is derivable.

In the second case $FV(\lambda x.B) = G - \{x\}$ and $G < (G - \{x\}), x$, hence $FV(B) < FV(\lambda x.B), x$. Further, $FV(\lambda x.B), x \vdash B$ is derivable (by Theorem 1.26), hence $FV(\lambda x.B) \vdash \lambda x.B$ is derivable.

In both cases $FV(\lambda x.B) = O_{\lambda x}(FV(B)) \leq O_{\lambda x}(\Gamma, x) = \Gamma$.

Case 4. A is $Wx \circ B$. Then Γ has the form Δ, x and $\Delta \vdash B$ is derivable by Generation lemma. By induction hypothesis $FV(B) \vdash B$ is derivable and $FV(B) \leq \Delta$, hence $FV(B), x \vdash Wx \circ B$ is derivable and $FV(B), x \leq \Delta, x$. Further, $FV(Wx \circ B) = FV(B), x$, hence $FV(Wx \circ B) \vdash Wx \circ B$ is derivable and $FV(Wx \circ B) \leq \Gamma$.

Case 5. A has the form $[B_1/x] \circ B_2$. Use Lemma 2.6.

Case 6. A has the form $\{yx\} \circ B$. Use Lemma 2.7.

Case 7. A has the form $S_x \circ B$. Use Lemma 2.8.

□

<i>(Beta)</i>	$(\lambda x.A)B \rightarrow [B/x] \circ A$	
<i>(App)</i>	$S \circ AB \rightarrow (S \circ A)(S \circ B)$	
<i>(Lambda)</i>	$S \circ \lambda x.A \rightarrow \lambda x.S_x \circ A$	
<i>(Var)</i>	$[B/x] \circ x \rightarrow B$	
<i>(Shift)</i>	$[B/x] \circ Wx \circ A \rightarrow A$	
<i>(Shift')</i>	$[B/x] \circ z \rightarrow z$	$(x \neq z)$
<i>(IdVar)</i>	$\{yx\} \circ x \rightarrow y$	
<i>(IdShift)</i>	$\{yx\} \circ Wx \circ A \rightarrow Wy \circ A$	
<i>(IdShift')</i>	$\{yx\} \circ z \rightarrow Wy \circ z$	$(x \neq z)$
<i>(LiftVar)</i>	$S_x \circ x \rightarrow x$	
<i>(LiftShift)</i>	$S_x \circ Wx \circ A \rightarrow Wx \circ S \circ A$	
<i>(LiftShift')</i>	$S_x \circ z \rightarrow Wx \circ S \circ z$	$(x \neq z)$
<i>(W)</i>	$Wx \circ z \rightarrow z$	$(x \neq z)$
<i>(α)</i>	$\lambda x.A \rightarrow \lambda y.\{yx\} \circ A$	$(*)$

Here $(*)$ is the condition $x \in FV(\lambda x.A) \ \& \ y \notin FV(\lambda x.A)$

Figure 3: The calculus $\lambda\alpha$

3 The calculus $\lambda\alpha$

Definition 3.1. The calculus $\lambda\alpha$ is shown in Figure 3 and Figure 4.

The strange names *IdVar* and *IdShift* will become clear later (I hope). Roughly speaking, $\{yx\}$ is similar to *id*, it does nothing except renaming bound variables.

The meaning of the rule *W* is as follows: if $\Gamma, x \vdash Wx \circ z$ is derivable, then $Wx \circ z$ denote the rightmost z in Γ . But if $x \neq z$, the rightmost z in Γ is the same as the rightmost z in Γ, x . Hence $\Gamma, x \vdash z$ is the same. The idea is not new, see [2] for example.

The rules *Shift'*, *IdShift'*, and *LiftShift'* provide confluence in the fol-

$$\begin{array}{c}
\frac{A \rightarrow A'}{\lambda x. A \rightarrow \lambda x. A'} \\
\\
\frac{A \rightarrow A'}{AB \rightarrow A'B} \qquad \frac{B \rightarrow B'}{AB \rightarrow AB'} \\
\\
\frac{S \rightarrow S'}{S \circ A \rightarrow S' \circ A} \qquad \frac{A \rightarrow A'}{S \circ A \rightarrow S \circ A'} \\
\\
\frac{B \rightarrow B'}{[B/x] \rightarrow [B'/x]} \\
\\
\frac{S \rightarrow S'}{S_x \rightarrow S'_x}
\end{array}$$

Figure 4: Compatible closure

lowing cases:

$$\begin{array}{c}
[B/x] \circ W_x \circ z \xrightarrow{Shift} z \\
\downarrow w \\
[B/x] \circ z
\end{array}$$

$$\begin{array}{c}
\{yx\} \circ W_x \circ z \xrightarrow{IdShift} W_y \circ z \\
\downarrow w \\
\{yx\} \circ z
\end{array}$$

$$\begin{array}{ccc}
S_x \circ W_x \circ z & \xrightarrow{LiftShift} & W_x \circ S \circ z \\
\downarrow W & & \\
S_x \circ z & &
\end{array}$$

Example 3.2.

$$\begin{array}{ll}
(\lambda x. \lambda y. x) y & \\
\rightarrow [y/x] \circ \lambda y. x & (Beta) \\
\rightarrow \lambda y. [y/x]_y \circ x & (Lambda) \\
\rightarrow \lambda y. W y \circ [y/x] \circ x & (LiftShift') \\
\rightarrow \lambda y. W y \circ y & (Var) \\
\rightarrow \lambda z. \{zy\} \circ W y \circ y & (\alpha) \\
\rightarrow \lambda z. W z \circ y & (IdShift) \\
\rightarrow \lambda z. y & (W)
\end{array}$$

Example 3.3.

$$\begin{array}{ll}
(\lambda x y z. x z (y z)) (\lambda x y. x) & \\
\rightarrow [\lambda x y. x/x] \circ \lambda y z. x z (y z) & (Beta) \\
\rightarrow \lambda y. [\lambda x y. x/x]_y \circ \lambda z. x z (y z) & (Lambda) \\
\rightarrow \lambda y z. [\lambda x y. x/x]_{yz} \circ x z (y z) & (Lambda) \\
\rightarrow \lambda y z. (\underline{[\lambda x y. x/x]_{yz} \circ x z}) ([\lambda x y. x/x]_{yz} \circ y z) & (App) \\
\Rightarrow \lambda y z. (\lambda y. z) (\underline{[\lambda x y. x/x]_{yz} \circ y z}) & (Example 3.4) \\
\Rightarrow \lambda y z. (\lambda y. z) (y z) & (Example 3.5) \\
\rightarrow \lambda y z. [y z/y] \circ z & (Beta) \\
\rightarrow \lambda y z. z & (Shift')
\end{array}$$

Example 3.4.

$$\begin{aligned}
& [\lambda xy.x/x]_{yz} \circ xz \\
& \rightarrow ([\lambda xy.x/x]_{yz} \circ x)(\underline{[\lambda xy.x/x]_{yz} \circ z}) & (App) \\
& \rightarrow ([\lambda xy.x/x]_{yz} \circ x) z & (LiftVar) \\
& \rightarrow (Wz \circ [\lambda xy.x/x]_y \circ x) z & (LiftShift') \\
& \rightarrow (Wz \circ Wy \circ [\lambda xy.x/x] \circ x) z & (LiftShift') \\
& \rightarrow (Wz \circ Wy \circ \lambda xy.x) z & (Var) \\
& \Rightarrow (\lambda xy.x) z & (\text{cause } \lambda xy.x \text{ is closed}) \\
& \rightarrow [z/x] \circ \lambda y.x & (Beta) \\
& \rightarrow \lambda y.[z/x]_y \circ x & (Lambda) \\
& \rightarrow \lambda y.Wy \circ [z/x] \circ x & (LiftShift') \\
& \rightarrow \lambda y.Wy \circ z & (Var) \\
& \rightarrow \lambda y.z & (W)
\end{aligned}$$

Example 3.5.

$$\begin{aligned}
& [\lambda xy.x/x]_{yz} \circ yz \\
& \rightarrow ([\lambda xy.x/x]_{yz} \circ y)(\underline{[\lambda xy.x/x]_{yz} \circ z}) & (App) \\
& \rightarrow ([\lambda xy.x/x]_{yz} \circ y) z & (LiftVar) \\
& \rightarrow (Wz \circ [\lambda xy.x/x]_y \circ y) z & (LiftShift') \\
& \rightarrow (Wz \circ y) z & (LiftVar) \\
& \rightarrow yz & (W)
\end{aligned}$$

Theorem 3.6. “Subject reduction”.

If $\Gamma \vdash A$ and $A \rightarrow B$ then $\Gamma \vdash B$.

Proof.

Case *Beta*.

$$\begin{array}{c}
\vdots \\
\Gamma, x \vdash A \\
\hline
\Gamma \vdash \lambda x.A
\end{array}
\quad
\begin{array}{c}
\vdots \\
\Gamma \vdash B
\end{array}
\quad
\begin{array}{c}
\vdots \\
\Gamma, x \vdash A
\end{array}$$

$$\begin{array}{c}
\Gamma \vdash (\lambda x.A)B
\end{array}
\quad
\begin{array}{c}
\vdots \\
\Gamma \vdash B \\
\hline
\Gamma \vdash [B/x] \triangleright \Gamma, x
\end{array}
\quad
\begin{array}{c}
\vdots \\
\Gamma, x \vdash A
\end{array}$$

$$\Gamma \vdash [B/x] \circ A$$

Case *App*.

$$\begin{array}{c}
\begin{array}{c} \vdots \\ \Gamma \vdash S \triangleright \Delta \end{array} \quad \frac{\begin{array}{c} \vdots \\ \Delta \vdash A \end{array} \quad \begin{array}{c} \vdots \\ \Delta \vdash B \end{array}}{\Delta \vdash AB} \\
\hline
\Gamma \vdash S \circ AB
\end{array}
\quad
\begin{array}{c}
\begin{array}{c} \vdots \\ \Gamma \vdash S \triangleright \Delta \end{array} \quad \begin{array}{c} \vdots \\ \Delta \vdash A \end{array} \\
\hline
\Gamma \vdash S \circ A
\end{array}
\quad
\begin{array}{c}
\begin{array}{c} \vdots \\ \Gamma \vdash S \triangleright \Delta \end{array} \quad \begin{array}{c} \vdots \\ \Delta \vdash B \end{array} \\
\hline
\Gamma \vdash S \circ B
\end{array} \\
\hline
\Gamma \vdash (S \circ A)(S \circ B)
\end{array}$$

Case *Lambda*.

$$\begin{array}{c}
\begin{array}{c} \vdots \\ \Gamma \vdash S \triangleright \Delta \end{array} \quad \frac{\begin{array}{c} \vdots \\ \Delta, x \vdash A \end{array}}{\Delta \vdash \lambda x. A} \\
\hline
\Gamma \vdash S \circ \lambda x. A
\end{array}
\quad
\begin{array}{c}
\frac{\begin{array}{c} \vdots \\ \Gamma \vdash S \triangleright \Delta \end{array}}{\Gamma, x \vdash S_x \triangleright \Delta, x} \quad \begin{array}{c} \vdots \\ \Delta, x \vdash A \end{array} \\
\hline
\frac{\Gamma, x \vdash S_x \circ A}{\Gamma \vdash \lambda x. S_x \circ A}
\end{array}$$

Case *Var*.

$$\begin{array}{c}
\begin{array}{c} \vdots \\ \Gamma \vdash B \end{array} \\
\hline
\Gamma \vdash [B/x] \triangleright \Gamma, x \quad \Gamma, x \vdash x \\
\hline
\Gamma \vdash [B/x] \circ x
\end{array}
\quad
\begin{array}{c} \vdots \\ \Gamma \vdash B \end{array}$$

Case *Shift*.

$$\begin{array}{c}
\begin{array}{c} \vdots \\ \Gamma \vdash B \end{array} \quad \frac{\Gamma, x \vdash W_x \triangleright \Gamma \quad \begin{array}{c} \vdots \\ \Gamma \vdash A \end{array}}{\Gamma, x \vdash W_x \circ A} \\
\hline
\Gamma \vdash [B/x] \triangleright \Gamma, x \quad \Gamma, x \vdash W_x \circ A \\
\hline
\Gamma \vdash [B/x] \circ W_x \circ A
\end{array}
\quad
\begin{array}{c} \vdots \\ \Gamma \vdash A \end{array}$$

Case *Shift'*.

$$\begin{array}{c}
\begin{array}{c} \vdots \\ \Gamma \vdash B \end{array} \quad \begin{array}{c} \vdots \\ \Gamma \vdash z \end{array} \\
\hline
\Gamma \vdash [B/x] \triangleright \Gamma, x \quad \Gamma, x \vdash z \\
\hline
\Gamma \vdash [B/x] \circ z
\end{array}
\quad
\begin{array}{c} \vdots \\ \Gamma \vdash z \end{array}$$

Case *IdVar*.

$$\frac{\Gamma, y \vdash \{yx\} \triangleright \Gamma, x \quad \Gamma, x \vdash x}{\Gamma, y \vdash \{yx\} \circ x} \quad \Gamma, y \vdash y$$

Case *IdShift*.

$$\frac{\Gamma, y \vdash \{yx\} \triangleright \Gamma, x \quad \frac{\Gamma, x \vdash Wx \triangleright \Gamma \quad \Gamma \vdash A}{\Gamma, x \vdash Wx \circ A}}{\Gamma, y \vdash \{yx\} \circ Wx \circ A} \quad \frac{\Gamma, y \vdash Wy \triangleright \Gamma \quad \Gamma \vdash A}{\Gamma, y \vdash Wy \circ A}$$

Case *IdShift'*.

$$\frac{\Gamma, y \vdash \{yx\} \triangleright \Gamma, x \quad \frac{\Gamma \vdash z}{\Gamma, x \vdash z}}{\Gamma, y \vdash \{yx\} \circ z} \quad \frac{\Gamma, y \vdash Wy \triangleright \Gamma \quad \Gamma \vdash z}{\Gamma, y \vdash Wy \circ z}$$

Case *LiftVar*.

$$\frac{\frac{\Gamma \vdash \dot{S} \triangleright \Delta}{\Gamma, x \vdash S_x \triangleright \Delta, x} \quad \Delta, x \vdash x}{\Gamma, x \vdash S_x \circ x} \quad \Gamma, x \vdash x$$

Case *LiftShift*.

$$\frac{\frac{\Gamma \vdash \dot{S} \triangleright \Delta}{\Gamma, x \vdash S_x \triangleright \Delta, x} \quad \frac{\Delta, x \vdash Wx \triangleright \Delta \quad \Delta \vdash A}{\Delta, x \vdash Wx \circ A}}{\Gamma, x \vdash S_x \circ Wx \circ A}$$

$$\frac{\Gamma, x \vdash Wx \triangleright \Gamma \quad \frac{\Gamma \vdash S \triangleright \Delta \quad \Delta \vdash A}{\Gamma \vdash S \circ A}}{\Gamma, x \vdash Wx \circ S \circ A}$$

Case *LiftShift'*.

$$\frac{\frac{\Gamma \vdash S \triangleright \Delta}{\Gamma, x \vdash S_x \triangleright \Delta, x} \quad \frac{\Delta \vdash z}{\Delta, x \vdash z}}{\Gamma, x \vdash S_x \circ z} \quad \frac{\Gamma, x \vdash Wx \triangleright \Gamma \quad \frac{\Gamma \vdash S \triangleright \Delta \quad \Delta \vdash z}{\Gamma \vdash S \circ z}}{\Gamma, x \vdash Wx \circ S \circ z}$$

Case *W*.

$$\frac{\Gamma, x \vdash Wx \triangleright \Gamma \quad \Gamma \vdash z}{\Gamma, x \vdash Wx \circ z} \quad \frac{\Gamma \vdash z}{\Gamma, x \vdash z}$$

Case α .

$$\frac{\Gamma, x \vdash A}{\Gamma \vdash \lambda x. A} \quad \frac{\Gamma, y \vdash \{yx\} \triangleright \Gamma, x \quad \Gamma, x \vdash A}{\frac{\Gamma, y \vdash \{yx\} \circ A}{\Gamma \vdash \lambda y. \{yx\} \circ A}}$$

□

Corollary 3.7. *Reducts of well-formed terms are well-formed.*

Theorem 3.8. *If A is a well-formed term and $A \rightarrow B$, then $FV(A) \geq FV(B)$.*

Proof.

Case *Beta*.

$$\begin{aligned} & FV([B/x] \circ A) \\ &= FV((\lambda x. A)B) \end{aligned}$$

Case *App*.

$$\begin{aligned}
& FV((S \circ A)(S \circ B)) \\
&= FV(S \circ A) \sqcup FV(S \circ B) \\
&= FV(S \circ AB)
\end{aligned}
\tag{Lemma 3.9}$$

Case *Lambda*.

$$\begin{aligned}
& FV(\lambda x. S_x \circ A) \\
&= O_{\lambda x}(FV(S_x \circ A)) \\
&= O_{\lambda x}(FV(W_x \circ S \circ \lambda x. A)) \\
&= O_{\lambda x}(FV(S \circ \lambda x. A), x) \\
&= FV(S \circ \lambda x. A)
\end{aligned}$$

Case *Var*.

$$\begin{aligned}
& FV([B/x] \circ x) \\
&= FV((\lambda x. x)B) \\
&= FV(\lambda x. x) \sqcup FV(B) \\
&= O_{\lambda x}(FV(x)) \sqcup FV(B) \\
&= O_{\lambda x}(\{x\}) \sqcup FV(B) \\
&= \emptyset \sqcup FV(B) \\
&= FV(B)
\end{aligned}$$

Case *Shift*.

$$\begin{aligned}
& FV([B/x] \circ W_x \circ A) \\
&= FV((\lambda x. W_x \circ A)B) \\
&= FV(\lambda x. W_x \circ A) \sqcup FV(B) \\
&\geq FV(\lambda x. W_x \circ A) \\
&= O_{\lambda x}(FV(W_x \circ A)) \\
&= O_{\lambda x}(FV(A), x) \\
&= FV(A)
\end{aligned}$$

Case *Shift'*.

$$\begin{aligned}
& FV([B/x] \circ z) \\
&= FV((\lambda x.z)B) \\
&= FV(\lambda x.z) \sqcup FV(B) \\
&\geq FV(\lambda x.z) \\
&= O_{\lambda x}(FV(z)) \\
&= O_{\lambda x}(\{z\}) \\
&= \{z\} \\
&= FV(z)
\end{aligned}$$

Case *IdVar*.

$$\begin{aligned}
& FV(\{yx\} \circ x) \\
&= FV(Wy \circ \lambda x.x) \\
&= FV(\lambda x.x), y \\
&> \{y\} \\
&= FV(y)
\end{aligned}$$

Case *IdShift*.

$$\begin{aligned}
& FV(\{yx\} \circ Wx \circ A) \\
&= FV(Wy \circ \lambda x.Wx \circ A) \\
&= FV(Wy \circ A) \quad (\text{because } FV(\lambda x.Wx \circ A) = FV(A))
\end{aligned}$$

Case *IdShift'*.

$$\begin{aligned}
& FV(\{yx\} \circ z) \\
&= FV(Wy \circ \lambda x.z) \\
&= FV(Wy \circ z) \quad (\text{because } FV(\lambda x.z) = FV(z) \text{ if } x \neq z)
\end{aligned}$$

Case *LiftVar*.

$$\begin{aligned}
& FV(S_x \circ x) \\
&= FV(Wx \circ S \circ \lambda x.x) \\
&= FV(S \circ \lambda x.x), x \\
&> \{x\} \\
&= FV(x)
\end{aligned}$$

Case *LiftShift*.

$$\begin{aligned}
& FV(S_x \circ W_x \circ A) \\
&= FV(W_x \circ S \circ \lambda x. W_x \circ A) \\
&= FV(W_x \circ S \circ A) \quad (\text{because } FV(\lambda x. W_x \circ A) = FV(A))
\end{aligned}$$

Case *LiftShift'*.

$$\begin{aligned}
& FV(S_x \circ z) \\
&= FV(W_x \circ S \circ \lambda x. z) \\
&= FV(W_x \circ S \circ z) \quad (\text{because } FV(\lambda x. z) = FV(z) \text{ if } x \neq z)
\end{aligned}$$

Case *W*.

$$\begin{aligned}
& FV(W_x \circ z) \\
&= \{z\}, x \\
&> \{z\} \\
&= FV(z)
\end{aligned}$$

Case α .

$$\begin{aligned}
& FV(\lambda y. \{yx\} \circ A) \\
&= O_{\lambda y}(FV(\{yx\} \circ A)) \\
&= O_{\lambda y}(FV(Wy \circ \lambda x. A)) \\
&= O_{\lambda y}(FV(\lambda x. A), y) \\
&= FV(\lambda x. A)
\end{aligned}$$

□

Lemma 3.9. *If $FV(S \circ AB)$ is defined, then $FV(S \circ AB) = FV(S \circ A) \sqcup FV(S \circ B)$*

Proof. Induction over the structure of S .

Case 1. S is $[C/x]$.

$$\begin{aligned}
& FV([C/x] \circ AB) \\
&= FV((\lambda x.AB)C) \\
&= FV(\lambda x.AB) \sqcup FV(C) \\
&= O_{\lambda x}(FV(AB)) \sqcup FV(C) \\
&= O_{\lambda x}(FV(A) \sqcup FV(B)) \sqcup FV(C) \\
&= O_{\lambda x}(FV(A)) \sqcup O_{\lambda x}(FV(B)) \sqcup FV(C) \quad (\text{Lemma 2.4}) \\
&= FV(\lambda x.A) \sqcup FV(\lambda x.B) \sqcup FV(C) \\
&= FV((\lambda x.A)C) \sqcup FV((\lambda x.B)C) \\
&= FV([C/x] \circ A) \sqcup FV([C/x] \circ B)
\end{aligned}$$

Case 2. S is Wx .

$$\begin{aligned}
& FV(Wx \circ AB) \\
&= FV(AB), x \\
&= (FV(A) \sqcup FV(B)), x \\
&= (FV(A), x) \sqcup (FV(B), x) \\
&= FV(Wx \circ A) \sqcup FV(Wx \circ B)
\end{aligned}$$

Case 3. S is $\{yx\}$.

$$\begin{aligned}
& FV(\{yx\} \circ AB) \\
&= FV(Wy \circ \lambda x.AB) \\
&= FV(\lambda x.AB), y \\
&= O_{\lambda x}(FV(AB)), y \\
&= O_{\lambda x}(FV(A) \sqcup FV(B)), y \\
&= (O_{\lambda x}(FV(A)) \sqcup O_{\lambda x}(FV(B))), y \quad (\text{Lemma 2.4}) \\
&= (O_{\lambda x}(FV(A)), y) \sqcup (O_{\lambda x}(FV(B)), y) \\
&= (FV(\lambda x.A), y) \sqcup (FV(\lambda x.B), y) \\
&= FV(Wy \circ \lambda x.A) \sqcup FV(Wy \circ \lambda x.B) \\
&= FV(\{yx\} \circ A) \sqcup FV(\{yx\} \circ B)
\end{aligned}$$

Case 4. S is S'_x .

$$\begin{aligned}
& FV(S'_x \circ AB) \\
&= FV(W_x \circ S' \circ \lambda x. AB) \\
&= FV(S' \circ \lambda x. AB), x \\
&= FV(S' \circ (\lambda x. A)(\lambda x. B)), x \quad (\text{cause } FV(\lambda x. AB) = FV((\lambda x. A)(\lambda x. B))) \\
&= (FV(S' \circ \lambda x. A) \sqcup FV(S' \circ \lambda x. B)), x \quad (\text{by the induction hypothesis}) \\
&= (FV(S' \circ \lambda x. A), x) \sqcup (FV(S' \circ \lambda x. B), x) \\
&= FV(W_x \circ S' \circ \lambda x. A) \sqcup FV(W_x \circ S' \circ \lambda x. B) \\
&= FV(S'_x \circ A) \sqcup FV(S'_x \circ B)
\end{aligned}$$

□

Syntax. $\Lambda v'$ is the set of terms inductively defined by the following BNF:

$$\begin{aligned}
 x &::= x \mid y \mid z \mid \dots && \text{(Variables)} \\
 a, b &::= x \mid \underline{1} \mid ab \mid \lambda a \mid a[s] && \text{(Terms)} \\
 s &::= b/ \mid \uparrow \mid id \mid \uparrow s && \text{(Substitutions)}
 \end{aligned}$$

Rewrite rules.

$$\begin{array}{ll}
 \textit{Beta} & (\lambda a)b \rightarrow a[b/] \\
 \textit{App} & (ab)[s] \rightarrow (a[s])(b[s]) \\
 \textit{Lambda} & (\lambda a)[s] \rightarrow \lambda(a[\uparrow s]) \\
 \textit{Var} & \underline{1}[b/] \rightarrow b \\
 \textit{Shift} & a[\uparrow][b/] \rightarrow a \\
 \textit{VarId} & \underline{1}[id] \rightarrow \underline{1} \\
 \textit{ShiftId} & a[\uparrow][id] \rightarrow a[\uparrow] \\
 \textit{VarLift} & \underline{1}[\uparrow s] \rightarrow \underline{1} \\
 \textit{ShiftLift} & a[\uparrow][\uparrow s] \rightarrow a[s][\uparrow]
 \end{array}$$

Figure 5: The calculus $\lambda v'$

4 The calculus $\lambda v'$

To prove confluence of $\lambda\alpha$, we consider the following calculus $\lambda v'$.

Definition 4.1. The calculus $\lambda v'$ is shown in Figure 5. This calculus contains both named variables and De Bruijn indices. There are no binders for named variables, they are free in all terms. By v' we denote $\lambda v'$ without *Beta*.

Proposition 4.2. *The calculus v' is terminating.*

Proof. The termination of v' is proved by a simple lexicographic ordering on two weights $\|\cdot\|_1$ and $\|\cdot\|_2$ defined on any terms or substitutions (see Figure 6).

$\ \times\ _1 = 2$	$\ \times\ _2 = 2$
$\ \underline{1}\ _1 = 2$	$\ \underline{1}\ _2 = 2$
$\ ab\ _1 = \ a\ _1 + \ b\ _1 + 1$	$\ ab\ _2 = \ a\ _2 + \ b\ _2 + 1$
$\ \lambda a\ _1 = \ a\ _1 + 1$	$\ \lambda a\ _2 = \ a\ _2 + 1$
$\ a[s]\ _1 = \ a\ _1 \cdot \ s\ _1$	$\ a[s]\ _2 = \ a\ _2 \cdot \ s\ _2$
$\ id\ _1 = 2$	$\ id\ _2 = 2$
$\ b/\ _1 = \ b\ _1$	$\ b/\ _2 = \ b\ _2$
$\ \uparrow\ _1 = 2$	$\ \uparrow\ _2 = 2$
$\ \uparrow s\ _1 = \ s\ _1$	$\ \uparrow s\ _2 = 2 \cdot \ s\ _2$

Figure 6: Interpretations for proving the termination of v'

$\|\cdot\|_1$ is strictly decreasing on all the rules but *ShiftLift*, on which it is decreasing. $\|\cdot\|_2$ is strictly decreasing on *ShiftLift*.

□

The calculus $\lambda v'$ is not locally confluent because of the presence of named variables. Now we define sets of well-formed terms and substitutions to prove confluence on these sets.

Definition 4.3. A *judgement* is an expression of the form $n \vdash a$ or of the form $n \vdash s \triangleright m$ ($n, m \in \mathbb{N}$). Inference rules for judgements are shown in Figure 7. A term a is *well-formed* iff $n \vdash a$ is derivable for some n .

Lemma 4.4. *Generation lemma.*

Each derivation of $n \vdash \times$ is an application of the rule R1, where n is 0.

Each derivation of $n \vdash \underline{1}$ is an application of the rule R2, where n is $m + 1$ for some m .

Each derivation of $n \vdash ab$ is an application of the rule R3 to some derivations of $n \vdash a$ and $n \vdash b$.

Each derivation of $n \vdash \lambda a$ is an application of the rule R4 to some derivation of $n + 1 \vdash a$.

Each derivation of $n \vdash a[s]$ is an application of the rule R5 to some derivations of $n \vdash s \triangleright m$ and $m \vdash a$ for some m .

$R1$	$0 \vdash x$
$R2$	$n + 1 \vdash \underline{1}$
$R3$	$\frac{n \vdash a \quad n \vdash b}{n \vdash ab}$
$R4$	$\frac{n + 1 \vdash a}{n \vdash \lambda a}$
$R5$	$\frac{n \vdash s \triangleright m \quad m \vdash a}{n \vdash a[s]}$
$R6$	$\frac{n \vdash b}{n \vdash b / \triangleright n + 1}$
$R7$	$n + 1 \vdash \uparrow \triangleright n$
$R8$	$n + 1 \vdash id \triangleright n + 1$
$R9$	$\frac{n \vdash s \triangleright m}{n + 1 \vdash \uparrow s \triangleright m + 1}$

Figure 7: Inference rules

Each derivation of $n \vdash b / \triangleright m$ is an application of the rule R6 to some derivation of $n \vdash b$, where m is $n + 1$.

Each derivation of $n \vdash \uparrow \triangleright m$ is an application of the rule R7, where n is $m + 1$.

Each derivation of $n \vdash id \triangleright m$ is an application of the rule R8, where n is $k + 1$ and m is $k + 1$ for some k .

Each derivation of $n \vdash \uparrow s \triangleright m$ is an application of the rule R9 to some derivation of $k \vdash s \triangleright l$, where n is $k + 1$ and m is $l + 1$.

Example 4.5. λx is not a well-formed term, but $\lambda(x[\uparrow])$ is well-formed. $\lambda\lambda x$ is not a well-formed term, but $\lambda\lambda(x[\uparrow][\uparrow])$ is well-formed.

$x[\underline{1}/]$ is not a well-formed term, but $x[\uparrow][\underline{1}/]$ is well-formed.

$x[b/]$ is not a well-formed term, but $x[\uparrow][b/]$ may be well-formed.

$x[id]$ is not a well-formed term (see the rule R8), but $x[\uparrow][id]$ is well-formed.

$x[\uparrow s]$ is not a well-formed term, but $x[\uparrow][\uparrow s]$ may be well-formed.

Corollary 4.6. *Subterms of well-formed terms are well-formed.*

Proposition 4.7. “Subject reduction”.

If $n \vdash a$ and $a \rightarrow b$, then $n \vdash b$.

Proof.

Case *Beta*.

$$\frac{\frac{\vdots}{n+1 \vdash a} \quad \vdots}{n \vdash \lambda a} \quad \vdots}{n \vdash (\lambda a)b} \quad \frac{\frac{\vdots}{n \vdash b} \quad \vdots}{n \vdash b / \triangleright n+1} \quad \vdots}{n \vdash a[b/]}$$

Case *App*.

$$\frac{\vdots}{n \vdash s \triangleright m} \quad \frac{\frac{\vdots}{m \vdash a} \quad \vdots}{m \vdash ab} \quad \vdots}{n \vdash (ab)[s]} \quad \frac{\frac{\vdots}{n \vdash s \triangleright m} \quad \vdots}{n \vdash a[s]} \quad \frac{\frac{\vdots}{n \vdash s \triangleright m} \quad \vdots}{n \vdash b[s]}}{n \vdash (a[s])(b[s])}$$

Case *Lambda*.

$$\begin{array}{c}
\begin{array}{c} \vdots \\ n \vdash s \triangleright m \end{array} \quad \frac{\begin{array}{c} \vdots \\ m+1 \vdash a \end{array}}{m \vdash \lambda a} \\
\hline
n \vdash (\lambda a)[s]
\end{array}
\quad
\begin{array}{c}
\frac{\begin{array}{c} \vdots \\ n \vdash s \triangleright m \end{array}}{n+1 \vdash \uparrow s \triangleright m+1} \quad \begin{array}{c} \vdots \\ m+1 \vdash a \end{array} \\
\hline
\frac{n+1 \vdash a[\uparrow s]}{n \vdash \lambda a[\uparrow s]}
\end{array}$$

Case *Var*.

$$\begin{array}{c}
\begin{array}{c} \vdots \\ n \vdash b \end{array} \\
\hline
n \vdash b / \triangleright n+1 \quad n+1 \vdash \underline{1} \\
\hline
n \vdash \underline{1}[b/]
\end{array}
\quad
\begin{array}{c} \vdots \\ n \vdash b \end{array}$$

Case *Shift*.

$$\begin{array}{c}
\begin{array}{c} \vdots \\ n \vdash b \end{array} \quad \frac{n+1 \vdash \uparrow \triangleright n \quad \begin{array}{c} \vdots \\ n \vdash a \end{array}}{n+1 \vdash a[\uparrow]} \\
\hline
n \vdash b / \triangleright n+1 \quad \frac{n+1 \vdash a[\uparrow]}{n \vdash a[\uparrow][b/]}
\end{array}
\quad
\begin{array}{c} \vdots \\ n \vdash a \end{array}$$

Case *VarId*.

$$\begin{array}{c}
n+1 \vdash id \triangleright n+1 \quad n+1 \vdash \underline{1} \\
\hline
n+1 \vdash \underline{1}[id]
\end{array}
\quad
n+1 \vdash \underline{1}$$

Case *ShiftId*.

$$\begin{array}{c}
n+1 \vdash id \triangleright n+1 \quad \frac{n+1 \vdash \uparrow \triangleright n \quad \begin{array}{c} \vdots \\ n \vdash a \end{array}}{n+1 \vdash a[\uparrow]} \\
\hline
n+1 \vdash a[\uparrow][id]
\end{array}
\quad
\frac{n+1 \vdash \uparrow \triangleright n \quad \begin{array}{c} \vdots \\ n \vdash a \end{array}}{n+1 \vdash a[\uparrow]}$$

Case *VarLift*.

$$\frac{\frac{\vdots}{n \vdash s \triangleright m} \quad \frac{n+1 \vdash \uparrow s \triangleright m+1 \quad m+1 \vdash \underline{1}}{n+1 \vdash \underline{1}[\uparrow s]}}{n+1 \vdash \underline{1}}$$

Case *ShiftLift*.

$$\frac{\frac{\vdots}{n \vdash s \triangleright m} \quad \frac{m+1 \vdash \uparrow \triangleright m \quad m \vdash a}{m+1 \vdash a[\uparrow]}}{n+1 \vdash a[\uparrow][\uparrow s]}$$

$$\frac{n+1 \vdash \uparrow \triangleright n \quad \frac{\vdots}{n \vdash s \triangleright m} \quad \frac{\vdots}{m \vdash a}}{n+1 \vdash a[s][\uparrow]}$$

□

Corollary 4.8. *Reducts of well-formed terms are well-formed.*

Lemma 4.9. *If a well-formed term “a” is an v' -normal form, then “a” does not contain substitutions of the forms $b/$, id , and $\uparrow s$.*

Proof. Induction over the structure of a .

Suppose, a contains a subterm $a'[b/]$, or $a'[id]$, or $a'[\uparrow s]$. By induction hypothesis, a' does not contain $[/]$, id , and \uparrow , hence a' has the form $c_1 c_2$, or λc , or $c[\uparrow]$, or $\underline{1}$ (by Generation lemma, a' can not be x). In each case we can apply some rewrite rule, hence a can not be an v' -normal form.

□

The following five lemmas have similar proofs, I prove the last lemma for example.

Lemma 4.10. *If $a[\uparrow(\uparrow)][\uparrow(b/)]$ is well-formed, there is a common v' -reduct of $a[\uparrow(\uparrow)][\uparrow(b/)]$ and a .*

Lemma 4.11. *If $a[\uparrow(\uparrow)][\uparrow(id)]$ is well-formed, there is a common v' -reduct of $a[\uparrow(\uparrow)][\uparrow(id)]$ and $a[\uparrow(\uparrow)]$.*

Lemma 4.12. *If $a[\uparrow(\uparrow)][\uparrow\uparrow s]$ and $a[\uparrow s][\uparrow(\uparrow)]$ are well-formed, there is a common v' -reduct of these terms.*

Lemma 4.13. *If $a[b/][s]$ and $a[\uparrow s][b[s]/]$ are well-formed, there is a common v' -reduct of these terms.*

Lemma 4.14. *If $a[id]$ is well-formed, there is a common v' -reduct of $a[id]$ and a .*

Proof. We prove the following stronger result: if $a[\uparrow^n id]$ ($n \geq 0$) is well-formed, there is a common v' -reduct of $a[\uparrow^n id]$ and a . The proof is by induction over the structure of a . By Lemma 4.9, we can assume that a does not contain $[/]$, id , and \uparrow .

Case 1. a has the form $a_1 a_2$.

$$(a_1 a_2)[\uparrow^n id] \xrightarrow{App} (a_1[\uparrow^n id])(a_2[\uparrow^n id])$$

Then we use the induction hypothesis.

Case 2. a has the form $\lambda a'$.

$$(\lambda a')[\uparrow^n id] \xrightarrow{Lambda} \lambda(a'[\uparrow^{n+1} id])$$

Then we use the induction hypothesis.

Case 3. a is $\underline{1}$.

If $n = 0$, then

$$\underline{1}[id] \xrightarrow{VarId} \underline{1}$$

If $n = m + 1$, then

$$\underline{1}[\uparrow^{m+1} id] \xrightarrow{VarLift} \underline{1}$$

Case 4. a has the form $a'[\uparrow]$.

If $n = 0$, then

$$a'[\uparrow][id] \xrightarrow{ShiftId} a'[\uparrow]$$

If $n = m + 1$, then

$$a'[\uparrow][\uparrow^{m+1} id] \xrightarrow{ShiftLift} a'[\uparrow^m id][\uparrow]$$

Then we use the induction hypothesis.

Note that a can not be \times by Generation lemma.

□

Theorem 4.15. *The rewriting system v' is locally confluent (hence, confluent) on the set of well-formed terms.*

Proof. Straightforward checking, using Lemma 4.10, Lemma 4.11, and Lemma 4.12 in the following cases

$$\begin{array}{ccc}
 (\lambda a)[\uparrow][b/] & \xrightarrow{Shift} & \lambda a \\
 \downarrow \text{Lambda} & & \\
 (\lambda(a[\uparrow(\uparrow)]))[b/] & & \\
 \\
 (\lambda a)[\uparrow][id] & \xrightarrow{ShiftId} & (\lambda a)[\uparrow] \\
 \downarrow \text{Lambda} & & \\
 (\lambda(a[\uparrow(\uparrow)]))[id] & & \\
 \\
 (\lambda a)[\uparrow][\uparrow s] & \xrightarrow{ShiftLift} & (\lambda a)[s][\uparrow] \\
 \downarrow \text{Lambda} & & \\
 (\lambda(a[\uparrow(\uparrow)]))[\uparrow s] & &
 \end{array}$$

□

Lemma 4.16. *Let R and S be two relations defined on the same set X , R is confluent and strongly normalizing, and S verifying the diamond property:*

$$\begin{array}{ccc}
 f & \xrightarrow{S} & g \\
 \downarrow S & & \downarrow S \\
 h & \xrightarrow{S} & k
 \end{array}$$

Suppose moreover that the following diagram holds:

$$\begin{array}{ccc}
 f & \xrightarrow{S} & g \\
 R \downarrow & & \downarrow R^* \\
 h & \xrightarrow{R^*SR^*} & k
 \end{array}$$

Here R^* is the reflexive and transitive closure of R . Then the relation R^*SR^* is confluent.

Proof. See [1] (Lemma 4.5). □

We shall apply the lemma with the following data. We take the set of well-formed terms as X , v' as R , and $Beta \parallel$ as S , where $Beta \parallel$ is the obvious parallelization of $Beta$ defined by:

$$\begin{array}{ll}
 a \rightarrow a & s \rightarrow s \\
 \\
 \frac{a_1 \rightarrow a_2 \quad b_1 \rightarrow b_2}{(\lambda a_1)b_1 \rightarrow a_2[b_2/]} & \frac{a_1 \rightarrow a_2}{\lambda a_1 \rightarrow \lambda a_2} \\
 \\
 \frac{a_1 \rightarrow a_2 \quad b_1 \rightarrow b_2}{a_1 b_1 \rightarrow a_2 b_2} & \frac{a_1 \rightarrow a_2 \quad s_1 \rightarrow s_2}{a_1[s_1] \rightarrow a_2[s_2]} \\
 \\
 \frac{b_1 \rightarrow b_2}{b_1/ \rightarrow b_2/} & \frac{s_1 \rightarrow s_2}{\uparrow s_1 \rightarrow \uparrow s_2}
 \end{array}$$

Proposition 4.17. v' and $Beta \parallel$ satisfy the conditions of Lemma 4.16.

Proof. The strong confluence of $Beta \parallel$ is obvious since $Beta$ by itself is a left linear system with no critical pairs. Now we check the second diagram.

Case *App*. $f \equiv (ab)[s] \xrightarrow{v'} (a[s])(b[s]) \equiv h$. Then there are two cases:

1. $f \equiv (ab)[s] \xrightarrow{Beta \parallel} (a'b')[s'] \equiv g$ with $a \xrightarrow{Beta \parallel} a'$, $b \xrightarrow{Beta \parallel} b'$, and $s \xrightarrow{Beta \parallel} s'$.

Then by definition of $Beta \parallel$ we have $(a[s])(b[s]) \xrightarrow{Beta \parallel} (a'[s'])(b'[s']) \equiv k$. But also $g \xrightarrow{v'} k$.

2. $f \equiv ((\lambda a)b)[s] \xrightarrow{\text{Beta}\parallel} a'[b'][s'] \equiv g$ with $a \xrightarrow{\text{Beta}\parallel} a'$, $b \xrightarrow{\text{Beta}\parallel} b'$, and $s \xrightarrow{\text{Beta}\parallel} s'$. Then $h \equiv ((\lambda a)[s])(b[s])$. We must then take $h \xrightarrow{v'} (\lambda(a[\uparrow s]))(b[s]) \equiv h_1$. Then $h_1 \xrightarrow{\text{Beta}\parallel} a'[\uparrow s'][b'[s'/]] \equiv h_2$. Using Lemma 4.13, we check that $h_2 \xrightarrow{v'^*} k$ and $g \xrightarrow{v'^*} k$ for some k . This subcase is the only interesting one. The cases of all other rewrite rules are simple and similar to subcase 1. \square

Theorem 4.18. *The rewriting system $\lambda v'$ is confluent on the set of well-formed terms.*

Proof. $\lambda v' \subseteq R^*SR^* \subseteq \lambda v'^*$. \square

5 Confluence

Recall that each derivable judgement has a unique derivation (Proposition 1.21).

Definition 5.1. For each derivable judgement $\Gamma \vdash A$ we put in correspondence some $\lambda v'$ -term as shown in Figure 8 (by recursion over the unique derivation of $\Gamma \vdash A$). For each derivable judgement $\Gamma \vdash S \triangleright \Delta$ we put in correspondence some $\lambda v'$ -substitution as shown in Figure 8.

$\Gamma \vdash A \longrightarrow a$ is shorthand for “ a corresponds to $\Gamma \vdash A$ ”

$\Gamma \vdash S \triangleright \Delta \longrightarrow s$ is shorthand for “ s corresponds to $\Gamma \vdash S \triangleright \Delta$ ”

Example 5.2.

$$\frac{\emptyset, x \vdash x \longrightarrow \underline{1}}{\emptyset \vdash \lambda x.x \longrightarrow \lambda \underline{1}}$$

Example 5.3.

$$\frac{\frac{\frac{\{x\} \vdash x \longrightarrow x}{\{x\}, y \vdash x \longrightarrow x[\uparrow]} \quad \{x\}, y \vdash y \longrightarrow \underline{1}}{\{x\}, y \vdash xy \longrightarrow (x[\uparrow])\underline{1}}}{\{x\} \vdash \lambda y.xy \longrightarrow \lambda(x[\uparrow])\underline{1}}$$

Example 5.4.

$$\frac{\{x\}, x \vdash Wx \triangleright \{x\} \longrightarrow \uparrow \quad \{x\} \vdash x \longrightarrow x}{\frac{\{x\}, x \vdash Wx \circ x \longrightarrow x[\uparrow]}{\{x\} \vdash \lambda x.Wx \circ x \longrightarrow \lambda(x[\uparrow])}}$$

Example 5.5.

$$\frac{\emptyset, y \vdash \{yx\} \triangleright \emptyset, x \longrightarrow id \quad \emptyset, x \vdash x \longrightarrow \underline{1}}{\frac{\emptyset, y \vdash \{yx\} \circ x \longrightarrow \underline{1}[id]}{\emptyset \vdash \lambda y.\{yx\} \circ x \longrightarrow \lambda(\underline{1}[id])}}$$

Proposition 5.6. *If $\Gamma \vdash A \longrightarrow a$, then a is well-formed. If $\Gamma \vdash S \triangleright \Delta \longrightarrow s$, then s is well-formed.*

Proof. Easy induction over the structure of derivations shows that if $\Gamma \vdash A \longrightarrow a$, then $n \vdash a$ is derivable, where n is the length of local part of Γ . Similarly, if $\Gamma \vdash S \triangleright \Delta \longrightarrow s$, then $n \vdash s \triangleright m$ is derivable, where n is the length of local part of Γ and m is the length of local part of Δ . □

$G \vdash x \longrightarrow x$	$(x \in G)$
$\Gamma, x \vdash x \longrightarrow \underline{1}$	
$\frac{\Gamma \vdash x \longrightarrow a}{\Gamma, y \vdash x \longrightarrow a[\uparrow]}$	$(x \neq y)$
$\frac{\Gamma \vdash A \longrightarrow a \quad \Gamma \vdash B \longrightarrow b}{\Gamma \vdash AB \longrightarrow ab}$	
$\frac{\Gamma, x \vdash A \longrightarrow a}{\Gamma \vdash \lambda x. A \longrightarrow \lambda a}$	
$\frac{\Gamma \vdash S \triangleright \Delta \longrightarrow s \quad \Delta \vdash A \longrightarrow a}{\Gamma \vdash S \circ A \longrightarrow a[s]}$	
$\frac{\Gamma \vdash B \longrightarrow b}{\Gamma \vdash [B/x] \triangleright \Gamma, x \longrightarrow b/}$	
$\Gamma, x \vdash Wx \triangleright \Gamma \longrightarrow \uparrow$	
$\Gamma, y \vdash \{yx\} \triangleright \Gamma, x \longrightarrow id$	
$\frac{\Gamma \vdash S \triangleright \Delta \longrightarrow s}{\Gamma, x \vdash S_x \triangleright \Delta, x \longrightarrow \uparrow\uparrow s}$	

Figure 8: Correspondence

Example 5.7.

$$\frac{\frac{\{x\}, x \vdash Wx \triangleright \{x\} \quad \{x\} \vdash x}{\{x\}, x \vdash Wx \circ x}}{\{x\} \vdash \lambda x. Wx \circ x} \quad \frac{\frac{1 \vdash \uparrow \triangleright 0 \quad 0 \vdash x}{1 \vdash x[\uparrow]}}{0 \vdash \lambda(x[\uparrow])}$$

Definition 5.8. We denote by σ the calculus $\lambda\alpha$ without the rules *Beta* and α .

Lemma 5.9. *If the following conditions hold*

- $\Gamma \vdash A \longrightarrow a$
- $A \xrightarrow{W} B$

then $\Gamma \vdash B \longrightarrow a$

Proof. If A contains a W -redex $Wx \circ z$ ($x \neq z$), then the unique derivation of $\Gamma \vdash A$ contains a sub-derivation of the form

$$\frac{\Delta, x \vdash Wx \triangleright \Delta \quad \begin{array}{c} \vdots \\ \Delta \vdash z \end{array}}{\Delta, x \vdash Wx \circ z}$$

and the unique derivation of $\Gamma \vdash B$ contains instead of it the sub-derivation

$$\frac{\begin{array}{c} \vdots \\ \Delta \vdash z \end{array}}{\Delta, x \vdash z}$$

Suppose $\Delta \vdash z \longrightarrow a'$, then $\Delta, x \vdash Wx \circ z \longrightarrow a'[\uparrow]$ and $\Delta, x \vdash z \longrightarrow a'[\uparrow]$

□

Lemma 5.10. *If the following conditions hold*

- $\Gamma \vdash A \longrightarrow a$
- $A \xrightarrow{\sigma \cup \{Beta\}} B$ and the rewrite rule is not W
- $\Gamma \vdash B \longrightarrow b$

then $a \xrightarrow{\lambda v'} b$.

Proof. The rules of $\sigma \cup \{Beta\}$ (except W) correspond to the rules of $\lambda v'$. For example, consider the rule $Shift'$. Suppose $A \xrightarrow{Shift'} B$. Then A contains a redex $[C/x] \circ z$ ($x \neq z$). The derivation of $\Gamma \vdash A$ must contain a sub-derivation of the form

$$\frac{\frac{\frac{\vdots}{\Delta \vdash C}}{\Delta \vdash [C/x] \triangleright \Delta, x} \quad \frac{\frac{\vdots}{\Delta \vdash z}}{\Delta, x \vdash z}}{\Delta \vdash [C/x] \circ z}$$

The derivation of $\Gamma \vdash B$ contains instead of it the sub-derivation

$$\frac{\vdots}{\Delta \vdash z}$$

Suppose $\Delta \vdash C \longrightarrow c$ and $\Delta \vdash z \longrightarrow a'$. Then
 $\Delta \vdash [C/x] \circ z \longrightarrow a'[\uparrow][c/]$

$$\frac{\frac{\frac{\vdots}{\Delta \vdash C \longrightarrow c}}{\Delta \vdash [C/x] \triangleright \Delta, x \longrightarrow c/} \quad \frac{\frac{\vdots}{\Delta \vdash z \longrightarrow a'}}{\Delta, x \vdash z \longrightarrow a'[\uparrow]}}{\Delta \vdash [C/x] \circ z \longrightarrow a'[\uparrow][c/]}$$

We obtain

$a'[\uparrow][c/] \xrightarrow{Shift} a'$, hence $a \xrightarrow{Shift} b$

□

Corollary 5.11. *If the following conditions hold*

- $\Gamma \vdash A \longrightarrow a$
- $A \xrightarrow{\sigma \cup \{Beta\}} B$
- $\Gamma \vdash B \longrightarrow b$

then $a \xrightarrow{\lambda v'} b$.

Lemma 5.12. *If the following conditions hold*

- $\Gamma \vdash B \longrightarrow b$
- $b \xrightarrow{\lambda v'} c$

then there exists a term C such that

- $\Gamma \vdash C \longrightarrow c$
- $B \xrightarrow{\sigma \cup \{Beta\}} C$

Proof. The rules of $\lambda v'$ corresponds to the rules of $\sigma \cup \{Beta\}$ except W . □

Corollary 5.13. *If the following conditions hold*

- $\Gamma \vdash B \longrightarrow b$
- $b \xrightarrow{\lambda v'} c$

then there exists a term C such that

- $\Gamma \vdash C \longrightarrow c$
- $B \xrightarrow{\sigma \cup \{Beta\}} C$

Definition 5.14. We write $A \equiv_{\Gamma} B$ iff

- $\Gamma \vdash A \longrightarrow a$
- $\Gamma \vdash B \longrightarrow a$

for some a .

Note that if $\Gamma \vdash A \longrightarrow a$, then $\Gamma \vdash A$ is derivable. Hence $A \equiv_{\Gamma} B$ imply A and B are well-formed.

Example 5.15. If $\Gamma = \{x\}, y$, then $Wy \circ x \equiv_{\Gamma} x$. Both terms correspond to $x[\uparrow]$.

Example 5.16. $\lambda x.Wx \circ x \equiv_{\{x\}} \lambda y.Wy \circ x \equiv_{\{x\}} \lambda y.x$

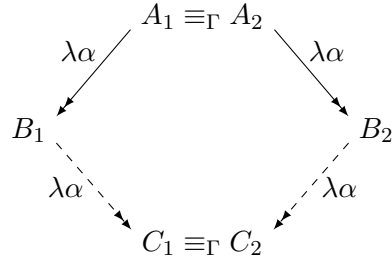
All these terms correspond to $\lambda(x[\uparrow])$. But $\lambda x.Wx \circ x \not\equiv_{\{x\}} \lambda x.x$, because $\lambda x.x$ corresponds to $\lambda \underline{1}$.

Theorem 5.17. *The calculus $\lambda\alpha$ is confluent in the following sense. Suppose*

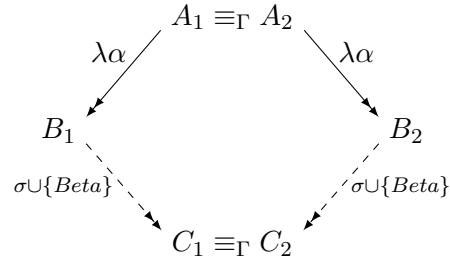
- $A_1 \equiv_{\Gamma} A_2$
- $A_1 \xrightarrow{\lambda\alpha} B_1$
- $A_2 \xrightarrow{\lambda\alpha} B_2$

then there are terms C_1 and C_2 such that

- $B_1 \xrightarrow{\lambda\alpha} C_1$
- $B_2 \xrightarrow{\lambda\alpha} C_2$
- $C_1 \equiv_{\Gamma} C_2$



Proof. We prove the following stronger result

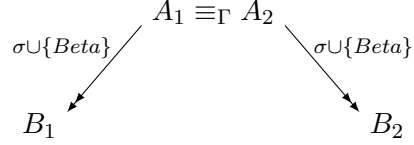


Suppose

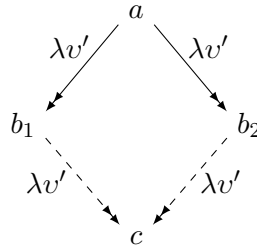
- $\Gamma \vdash A_1 \longrightarrow a$
- $\Gamma \vdash A_2 \longrightarrow a$
- $\Gamma \vdash B_1 \longrightarrow b_1$

- $\Gamma \vdash B_2 \longrightarrow b_2$

Case 1.

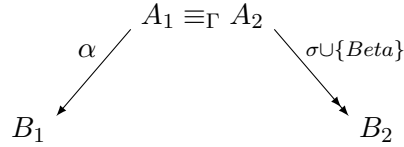


Using Corollary 5.11 and confluence of $\lambda v'$ on the set of well-formed terms, we obtain



Then we use Corollary 5.13.

Case 2.

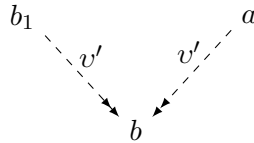


The term A_1 contains an α -redex of the form $\lambda x.A'$. The term B_1 contains instead of it a subterm $\lambda y.\{yx\} \circ A'$. The derivation of $\Gamma \vdash A_1$ contains a sub-derivation of the form $\Delta \vdash \lambda x.A'$.

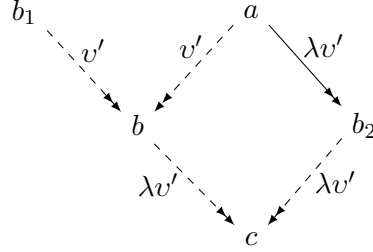
Suppose $\Delta \vdash \lambda x.A' \longrightarrow \lambda a'$

Then $\Delta \vdash \lambda y.\{yx\} \circ A' \longrightarrow \lambda(a'[id])$

Using Lemma 4.14, we obtain

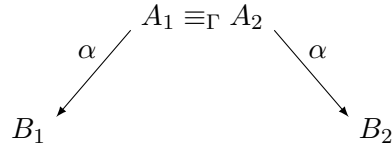


for some b , hence

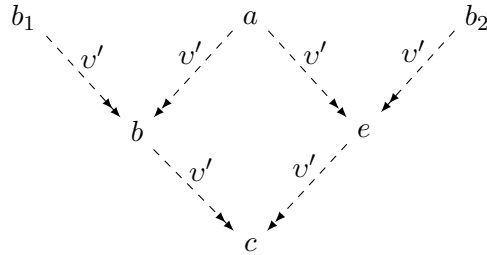


Then we use Corollary 5.13.

Case 3.



As in the previous case.



□

For good terms we can get a better result.

Definition 5.18. *A is a good term* iff there is a global context G such that $G \vdash A$ is derivable (the local context is empty).

Fact 5.19. *Each usual lambda-term is a good term.*

Proposition 5.20. *All reducts of good terms are good.*

Proof. By Theorem 3.6.

□

Example 5.21. A term of the form $Wx \circ A$ is not good, because if $\Gamma \vdash Wx \circ A$ is derivable then Γ must have the form Δ, x , hence its local part can not be empty.

$$\frac{\Delta, x \vdash Wx \triangleright \Delta \quad \begin{array}{c} \vdots \\ \Delta \vdash A \end{array}}{\Delta, x \vdash Wx \circ A}$$

But a term of the form $\lambda x. Wx \circ A$ can be good. In a good term, each symbol W must be “killed” by lambda or another binder.

Example 5.22. The term $\lambda x. \lambda y. Wy \circ Wx \circ z$ is good.

Definition 5.23. For good terms A and B we write $A \equiv_\alpha B$ iff

$A \equiv_{FV(A) \cup FV(B)} B$.

It is easy to prove that we can use any set G such that $FV(A) \cup FV(B) \subseteq G$ instead of $FV(A) \cup FV(B)$.

Example 5.24. $\lambda x. Wx \circ x \equiv_\alpha \lambda y. Wy \circ x \equiv_\alpha \lambda y. x$

But $\lambda x. Wx \circ x \not\equiv_\alpha \lambda x. x$

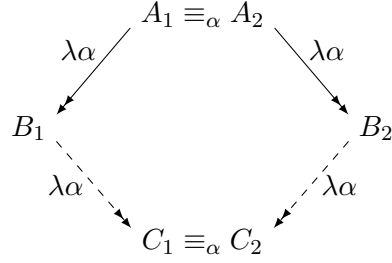
Fact 5.25. On the set Λ of usual lambda-terms the relation \equiv_α is the same as the usual α -congruence.

Corollary 5.26. Suppose

- A_1 and A_2 are good terms
- $A_1 \equiv_\alpha A_2$
- $A_1 \xrightarrow{\lambda\alpha} B_1$
- $A_2 \xrightarrow{\lambda\alpha} B_2$

then there are terms C_1 and C_2 such that

- $B_1 \xrightarrow{\lambda\alpha} C_1$
- $B_2 \xrightarrow{\lambda\alpha} C_2$
- $C_1 \equiv_\alpha C_2$



Definition 5.27. Really, it is convenient to use a slightly different notation, shown in Figure 9. We write $s \circ a$ instead of $a[s]$, W instead of \uparrow , and $[b/]$ instead of $b/$.

Example 5.28.

$$\frac{\frac{\{x\}, x \vdash Wx \triangleright \{x\} \longrightarrow W \quad \{x\} \vdash x \longrightarrow x}{\{x\}, x \vdash Wx \circ x \longrightarrow W \circ x}}{\{x\} \vdash \lambda x.Wx \circ x \longrightarrow \lambda W \circ x}$$

Example 5.29.

$$\frac{\frac{\emptyset, y \vdash \{yx\} \triangleright \emptyset, x \longrightarrow id \quad \emptyset, x \vdash x \longrightarrow \underline{1}}{\emptyset, y \vdash \{yx\} \circ x \longrightarrow id \circ \underline{1}}}{\emptyset \vdash \lambda y.\{yx\} \circ x \longrightarrow \lambda id \circ \underline{1}}$$

Example 5.30.

$$\frac{\frac{\{x, y\} \vdash x \longrightarrow x}{\{x, y\} \vdash [x/x] \triangleright \{x, y\}, x \longrightarrow [x/]} \quad \frac{\{x, y\} \vdash y \longrightarrow y}{\{x, y\}, x \vdash y \longrightarrow W \circ y}}{\{x, y\} \vdash [x/x] \circ y \longrightarrow [x/] \circ W \circ y}$$

$x ::= x \mid y \mid z \mid \dots$	(Variables)
$a, b ::= x \mid \underline{1} \mid ab \mid \lambda a \mid s \circ a$	(Terms)
$s ::= [b/] \mid W \mid id \mid \uparrow s$	(Substitutions)
$G \vdash x \longrightarrow x$	$(x \in G)$
$\Gamma, x \vdash x \longrightarrow \underline{1}$	
$\frac{\Gamma \vdash x \longrightarrow a}{\Gamma, y \vdash x \longrightarrow W \circ a}$	$(x \neq y)$
$\frac{\Gamma \vdash A \longrightarrow a \quad \Gamma \vdash B \longrightarrow b}{\Gamma \vdash AB \longrightarrow ab}$	
$\frac{\Gamma, x \vdash A \longrightarrow a}{\Gamma \vdash \lambda x. A \longrightarrow \lambda a}$	
$\frac{\Gamma \vdash S \triangleright \Delta \longrightarrow s \quad \Delta \vdash A \longrightarrow a}{\Gamma \vdash S \circ A \longrightarrow s \circ a}$	
$\frac{\Gamma \vdash B \longrightarrow b}{\Gamma \vdash [B/x] \triangleright \Gamma, x \longrightarrow [b/]}$	
$\Gamma, x \vdash Wx \triangleright \Gamma \longrightarrow W$	
$\Gamma, y \vdash \{yx\} \triangleright \Gamma, x \longrightarrow id$	
$\frac{\Gamma \vdash S \triangleright \Delta \longrightarrow s}{\Gamma, x \vdash S_x \triangleright \Delta, x \longrightarrow \uparrow s}$	

Figure 9: Correspondence

Syntax. The set of σ -normal forms is inductively defined by the following BNF:

$$\begin{array}{ll}
x, y, z ::= x \mid y \mid z \mid \dots & \text{(Variables)} \\
B ::= Wz \circ z \mid Wx \circ B & \text{(Blocks)} \\
A, B ::= x \mid B \mid AB \mid \lambda x. A & \text{(Terms)}
\end{array}$$

Figure 10: σ -normal forms

6 Normal forms

Lemma 6.1. *A well-formed term A is a σ -normal form iff it is constructed from variables and blocks of the form $Wx_1 \circ \dots \circ Wx_n \circ Wz \circ z$ ($n \geq 0$) by application and abstraction. See Figure 10.*

Proof. Induction over the structure of A . Suppose A has the form $S \circ B$. If B has the form $B_1 B_2$, we can apply the rule *App* and A can not be a σ -normal form. If B has the form $\lambda x. B'$, we can apply the rule *Lambda* and A can not be a σ -normal form. Hence, by induction hypothesis, B must be a variable or a block.

Case 1. B is a variable z , A is $S \circ z$. If S has the form $[C/x]$, we can apply the rule *Var* (if $x = z$) or the rule *Shift'* (if $x \neq z$). If S has the form $\{yx\}$, we can apply the rule *IdVar* or the rule *IdShift'*. If S has the form S'_x , we can apply the rule *LiftVar* or the rule *LiftShift'*. If S has the form Wx and $x \neq z$, we can apply the rule *W*. Hence, S must be Wz and A is $Wz \circ z$.

Case 2. B is a block and has the form $Wx \circ B'$, A is $S \circ Wx \circ B'$. If S has the form $[C/x]$, we can apply the rule *Shift*. If S has the form $\{yx\}$, we can apply the rule *IdShift*. If S has the form S'_x , we can apply the rule *LiftShift*. Hence, S must have the form Wy and A is the block $Wy \circ B$. \square

Theorem 6.2. *If A is a good term and a $\sigma \cup \{\alpha\}$ -normal form, then A is a usual lambda-term (i.e. without explicit substitutions).*

Proof. By the previous lemma, it is sufficient to prove that A does not contain blocks. Suppose A contains a block $Wx_1 \circ \dots \circ Wx_n \circ Wz \circ z$. By Generation lemma, the derivation of $G \vdash A$ contains a sub-derivation of a judgement

$\Gamma, z, x_n \dots x_1 \vdash Wx_1 \circ \dots Wx_n \circ Wz \circ z$ for some Γ . Below this judgement we use only the rules $R4$ and $R5$ from Figure 1. Note that

$$FV(Wx_1 \circ \dots Wx_n \circ Wz \circ z) = \{z\}, z, x_n \dots x_1$$

Suppose B_1 is a well-formed term, constructed from $Wx_1 \circ \dots Wx_n \circ Wz \circ z$ and something else by application. Then $FV(B_1)$ has the form $\Delta_1, z, x_n \dots x_1$ and $z \in \Delta_1$. After the first application of $R5$ we obtain

$$\frac{\begin{array}{c} \vdots \\ \Gamma, z, x_n \dots x_1 \vdash B_1 \end{array}}{\Gamma, z, x_n \dots x_2 \vdash \lambda x_1. B_1}$$

and $FV(\lambda x_1. B_1) = \Delta_1, z, x_n \dots x_2$.

Suppose B_2 is a well-formed term, constructed from $\lambda x_1. B_1$ and something else by application. Then $FV(B_2)$ has the form $\Delta_2, z, x_n \dots x_2$ and $z \in \Delta_2$. After the second application of $R5$ we obtain

$$\frac{\begin{array}{c} \vdots \\ \Gamma, z, x_n \dots x_2 \vdash B_2 \end{array}}{\Gamma, z, x_n \dots x_3 \vdash \lambda x_2. B_2}$$

and $FV(\lambda x_2. B_2) = \Delta_2, z, x_n \dots x_3$.

On the $n + 1$ -th application of $R5$ we have

$$\frac{\begin{array}{c} \vdots \\ \Gamma, z \vdash B_{n+1} \end{array}}{\Gamma \vdash \lambda z. B_{n+1}}$$

where $\lambda z. B_{n+1}$ is a subterm of A . $FV(B_{n+1})$ has the form Δ_{n+1}, z and $z \in \Delta_{n+1}$. But then $z \in FV(\lambda z. B_{n+1})$ and we can apply the rule α to $\lambda z. B_{n+1}$, hence A can not be a $\sigma \cup \{\alpha\}$ -normal form.

□

Syntax. $\Lambda v''$ is the set of terms inductively defined by the following BNF:

$$\begin{aligned}
x &::= x \mid y \mid z \mid \dots && \text{(Variables)} \\
a, b &::= x \mid \underline{1} \mid ab \mid \lambda a \mid \underline{\lambda} a \mid a[s] && \text{(Terms)} \\
s &::= b/ \mid \uparrow \mid id \mid \uparrow s && \text{(Substitutions)}
\end{aligned}$$

Rewrite rules.

$$\begin{array}{ll}
App & (ab)[s] \rightarrow (a[s])(b[s]) \\
Lambda & (\lambda a)[s] \rightarrow \lambda(a[\uparrow s]) \\
Lambda' & (\lambda a)[s] \rightarrow \underline{\lambda}(a[\uparrow s]) \\
Lambda'' & (\underline{\lambda} a)[s] \rightarrow \lambda(a[\uparrow s]) \\
Lambda''' & (\underline{\lambda} a)[s] \rightarrow \underline{\lambda}(a[\uparrow s]) \\
Var & \underline{1}[b/] \rightarrow b \\
Shift & a[\uparrow][b/] \rightarrow a \\
VarId & \underline{1}[id] \rightarrow \underline{1} \\
ShiftId & a[\uparrow][id] \rightarrow a[\uparrow] \\
VarLift & \underline{1}[\uparrow s] \rightarrow \underline{1} \\
ShiftLift & a[\uparrow][\uparrow s] \rightarrow a[s][\uparrow] \\
\alpha & \underline{\lambda} a \rightarrow \lambda(a[id]) \\
\xi & \underline{\lambda} a \rightarrow \lambda a
\end{array}$$

Figure 11: The calculus v''

7 $\sigma \cup \{\alpha\}$ is strongly normalizing

Definition 7.1. The calculus v'' is shown in Figure 11 (the same calculus in better notation is shown in Figure 12). It differs from v' by the presence of terms of new kind $\underline{\lambda} a$ and five new rewrite rules *Lambda'*, *Lambda''*, *Lambda'''*, α , and ξ .

Definition 7.2. For each derivable judgement $\Gamma \vdash A$ we put in correspondence

Syntax. $\Lambda v''$ is the set of terms inductively defined by the following BNF:

$$\begin{aligned}
 x &::= x \mid y \mid z \mid \dots && \text{(Variables)} \\
 a, b &::= x \mid \underline{1} \mid ab \mid \lambda a \mid \underline{\lambda} a \mid s \circ a && \text{(Terms)} \\
 s &::= [b/] \mid W \mid id \mid \uparrow s && \text{(Substitutions)}
 \end{aligned}$$

Rewrite rules.

<i>App</i>	$s \circ ab \rightarrow (s \circ a)(s \circ b)$
<i>Lambda</i>	$s \circ \lambda a \rightarrow \lambda \uparrow s \circ a$
<i>Lambda'</i>	$s \circ \lambda a \rightarrow \underline{\lambda} \uparrow s \circ a$
<i>Lambda''</i>	$s \circ \underline{\lambda} a \rightarrow \lambda \uparrow s \circ a$
<i>Lambda'''</i>	$s \circ \underline{\lambda} a \rightarrow \underline{\lambda} \uparrow s \circ a$
<i>Var</i>	$[b/] \circ \underline{1} \rightarrow b$
<i>Shift</i>	$[b/] \circ W \circ a \rightarrow a$
<i>IdVar</i>	$id \circ \underline{1} \rightarrow \underline{1}$
<i>IdShift</i>	$id \circ W \circ a \rightarrow W \circ a$
<i>LiftVar</i>	$\uparrow s \circ \underline{1} \rightarrow \underline{1}$
<i>LiftShift</i>	$\uparrow s \circ W \circ a \rightarrow W \circ s \circ a$
α	$\underline{\lambda} a \rightarrow \lambda id \circ a$
ξ	$\underline{\lambda} a \rightarrow \lambda a$

Figure 12: The calculus v''

some $\Lambda v''$ -term as shown in Figure 8 (or Figure 9), but with the following changes for abstraction:

$$\frac{\Gamma, x \vdash A \longrightarrow a}{\Gamma \vdash \lambda x. A \longrightarrow \lambda a} \quad (x \notin FV(\lambda x. A))$$

$$\frac{\Gamma, x \vdash A \longrightarrow a}{\Gamma \vdash \lambda x. A \longrightarrow \underline{\lambda} a} \quad (x \in FV(\lambda x. A))$$

Example 7.3.

$$\frac{\{x\}, x \vdash Wx \triangleright \{x\} \longrightarrow \uparrow \quad \{x\} \vdash x \longrightarrow x}{\{x\}, x \vdash Wx \circ x \longrightarrow x[\uparrow]} \quad \{x\} \vdash \lambda x. Wx \circ x \longrightarrow \underline{\lambda}(x[\uparrow])$$

Or, better

$$\frac{\{x\}, x \vdash Wx \triangleright \{x\} \longrightarrow W \quad \{x\} \vdash x \longrightarrow x}{\{x\}, x \vdash Wx \circ x \longrightarrow W \circ x} \quad \{x\} \vdash \lambda x. Wx \circ x \longrightarrow \underline{\lambda} W \circ x$$

Theorem 7.4. *If the following conditions hold*

- $A_0 \xrightarrow{\sigma \cup \{\alpha\}} A_1 \xrightarrow{\sigma \cup \{\alpha\}} \dots \xrightarrow{\sigma \cup \{\alpha\}} A_n \xrightarrow{\sigma \cup \{\alpha\}} \dots$
- $\Gamma \vdash A_n \longrightarrow a_n \quad \forall n \in \mathbb{N}$

then we get

$$a_0 \xrightarrow{v''} a_1 \xrightarrow{v''} \dots \xrightarrow{v''} a_n \xrightarrow{v''} \dots$$

Proof. Recall that the rule α in $\sigma \cup \{\alpha\}$ is as follows

(α) $\lambda x. A \rightarrow \lambda y. \{y x\} \circ A$ where $x \in FV(\lambda x. A)$ & $y \notin FV(\lambda x. A)$

It corresponds to the rule α of v'' .

By Theorem 3.8, $A \xrightarrow{\sigma \cup \{\alpha\}} B$ imply $FV(A) \geq FV(B)$, hence if $x \notin FV(\lambda x. A)$ then $x \notin FV(\lambda x. B)$. Hence λa can not go to $\underline{\lambda} b$ in the process of rewriting under lambda. □

Example 7.5.

$$\{x\} \vdash \lambda x. Wx \circ x \xrightarrow{\alpha} \lambda y. \{yx\} \circ Wx \circ x \xrightarrow{IdShift} \lambda y. Wy \circ x \xrightarrow{W} \lambda y. x$$

goes to

$$0 \vdash \underline{\lambda} W \circ x \xrightarrow{\alpha} \lambda id \circ W \circ x \xrightarrow{IdShift} \lambda W \circ x$$

Example 7.6.

$$\{x, y\} \vdash \lambda x. [Wx \circ x/x] \circ y \xrightarrow{Shift'} \lambda x. y$$

goes to

$$0 \vdash \underline{\lambda}[W \circ x/] \circ W \circ W \circ y \xrightarrow{Shift} \underline{\lambda} W \circ y \xrightarrow{\xi} \lambda W \circ y$$

Why so many W ? See

$$\frac{\frac{\{x, y\}, x \vdash Wx \triangleright \{x, y\} \quad \{x, y\} \vdash x}{\{x, y\}, x \vdash Wx \circ x} \quad \frac{\{x, y\} \vdash y}{\{x, y\}, x \vdash y}}{\frac{\{x, y\}, x \vdash [Wx \circ x/x] \triangleright \{x, y\}, x, x \quad \{x, y\}, x, x \vdash y}{\{x, y\}, x \vdash [Wx \circ x/x] \circ y}} \quad \frac{}{\{x, y\} \vdash \lambda x. [Wx \circ x/x] \circ y}$$

To prove that v'' is strongly normalizing, we use the method of semantic labelling. See [3].

Definition 7.7. To each term a and each substitution s we put in correspondence natural numbers (weights) $\|a\|$ and $\|s\|$ defined as follows:

$$\begin{aligned} \|\mathbf{x}\| &= 0 \\ \|\mathbf{1}\| &= 0 \\ \|ab\| &= \max(\|a\|, \|b\|) \\ \|\lambda a\| &= \|a\| + 1 \\ \|\underline{\lambda} a\| &= \|a\| + 1 \\ \|s \circ a\| &= \|s\| + \|a\| \\ \|[b/]\| &= \|b\| \\ \|W\| &= 0 \\ \|id\| &= 0 \\ \|\uparrow s\| &= \|s\| \end{aligned}$$

Note that all functional symbols of v'' (application, λ , $\underline{\lambda}$, \circ , $[/]$, and \uparrow) turn to monotone functions of \mathbb{N} to \mathbb{N} or of $\mathbb{N} \times \mathbb{N}$ to \mathbb{N} .

Syntax. $\Lambda v'''$ is the set of terms inductively defined by the following BNF:

$$\begin{aligned} x &::= x \mid y \mid z \mid \dots && \text{(Variables)} \\ a, b &::= x \mid \underline{1} \mid ab \mid \lambda a \mid \underline{\lambda}_i a \mid s \circ_i a && \text{(Terms)} \\ s &::= [b/] \mid W \mid id \mid \uparrow s && \text{(Substitutions)} \end{aligned}$$

Rewrite rules.

<i>App</i>	$s \circ_{\max(i,j)} ab \rightarrow (s \circ_i a)(s \circ_j b)$	
<i>Lambda</i>	$s \circ_{k+1} \lambda a \rightarrow \lambda \uparrow s \circ_k a$	
<i>Lambda'</i>	$s \circ_{k+1} \lambda a \rightarrow \underline{\lambda}_{k+1} \uparrow s \circ_k a$	
<i>Lambda''</i>	$s \circ_{i+j+1} \underline{\lambda}_{j+1} a \rightarrow \lambda \uparrow s \circ_{i+j} a$	
<i>Lambda'''</i>	$s \circ_{i+j+1} \underline{\lambda}_{j+1} a \rightarrow \underline{\lambda}_{i+j+1} \uparrow s \circ_{i+j} a$	
<i>Var</i>	$[b/] \circ_i \underline{1} \rightarrow b$	
<i>Shift</i>	$[b/] \circ_{i+j} W \circ_j a \rightarrow a$	
<i>IdVar</i>	$id \circ_0 \underline{1} \rightarrow \underline{1}$	
<i>IdShift</i>	$id \circ_i W \circ_i a \rightarrow W \circ_i a$	
<i>LiftVar</i>	$\uparrow s \circ_i \underline{1} \rightarrow \underline{1}$	
<i>LiftShift</i>	$\uparrow s \circ_{i+j} W \circ_j a \rightarrow W \circ_{i+j} s \circ_{i+j} a$	
α	$\underline{\lambda}_{i+1} a \rightarrow \lambda id \circ_i a$	
ξ	$\underline{\lambda}_i a \rightarrow \lambda a$	
<i>Decr₁</i>	$\underline{\lambda}_i a \rightarrow \underline{\lambda}_j a$	$(i > j)$
<i>Decr₂</i>	$s \circ_i a \rightarrow s \circ_j a$	$(i > j)$

Figure 13: The calculus v'''

Definition 7.8. The calculus v''' is shown in Figure 13. It differs from v'' by the presence of natural indexes in $\underline{\lambda}_i a$ and $s \circ_i a$. The rules of v''' are the rules of v'' , where all terms $\underline{\lambda} a$ and $s \circ a$ are labelled by their weights (there are also the new rules $Decr_1$ and $Decr_2$).

Theorem 7.9. v''' is strongly normalizing.

Proof. By choosing the well-founded precedence

$$\begin{aligned}
& \circ_i > \text{application} \\
& \circ_i > \circ_j \quad (i > j) \\
& \circ_i > \lambda \\
& \circ_i > \uparrow \\
& \circ_i > \underline{\lambda}_i \\
& \uparrow > W \\
& \underline{\lambda}_i > \lambda \\
& \underline{\lambda}_{i+1} > \circ_i \\
& \underline{\lambda}_i > id \\
& \underline{\lambda}_i > \underline{\lambda}_j \quad (i > j)
\end{aligned}$$

termination is easily proved by the lexicographic path order. □

Theorem 7.10. v'' is strongly normalizing.

Proof. For any infinite sequence

$$a_0 \xrightarrow{v''} a_1 \xrightarrow{v''} a_2 \xrightarrow{v''} \dots \xrightarrow{v''} a_n \xrightarrow{v''} \dots$$

we can get an infinite sequence

$$a'_0 \xrightarrow{v'''} a'_1 \xrightarrow{v'''} a'_2 \xrightarrow{v'''} \dots \xrightarrow{v'''} a'_n \xrightarrow{v'''} \dots$$

by labelling all subterms of the forms $\underline{\lambda} a$ and $s \circ a$ by their weights. See [3] (Theorem 81) for details. □

Theorem 7.11. $\sigma \cup \{\alpha\}$ is strongly normalizing on the set of well-formed terms.

Proof. By Theorem 7.4 and Theorem 7.10.

□

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